7.3 If $\mu \le 1$, then $\mathbb{P}(Z_n \to 0) = \mathbb{P}(\text{extinction}) = 1$

$\Rightarrow \mathbb{P}(\lim Z_n/\mu^n = 0) = \mathbb{P}(Z_n \to 0) = 1$ as well. – the statement is trivial. So $\boxed{\text{assume } \mu > 1}$

Let $Z_n^{(k)}$ be the number of descendants of the k-th child of the root in the ~~(n+1)~~ $(1+n)$-th generation, as in the hint.

For every k, this $\{Z_n^{(k)}\}_{n=0}^{\infty}$ is also a branching process with the same law as Z_n, so

$$\mathbb{P}\left(Z_n^{(k)}/\mu^n \to 0\right) = \mathbb{P}\left(Z_n/\mu^n \to 0\right) = \theta \quad \text{for each } k.$$

Now on the event $\{Z_1 = K\}$ we have

$$Z_n = \sum_{k=1}^{K} Z_{n-1}^{(k)} \Rightarrow \frac{Z_n}{\mu^n} = \frac{1}{\mu} \sum_{k=1}^{K} \frac{Z_{n-1}^{(k)}}{\mu^{n-1}}.$$

This is a sum of finitely many nonnegative ~~numbers~~ terms, so it goes to zero iff each term goes to zero. So $\boxed{\text{on the event } \{Z_1 = K\}}$ we have $\left\{\frac{Z_n}{\mu^n} \to 0\right\} = \left\{\frac{Z_{n-1}^{(k)}}{\mu^{n}} \to 0 \text{ for } k = 1, 2, \dots, K\right\}$

so $\mathbb{P}\left(\frac{Z_n}{\mu^n} \to 0 \,\middle|\, Z_1 = k\right) = \mathbb{P}\left(\frac{Z_{n-1}^{(k)}}{\mu^{n-1}} \to 0 \text{ for } k = 1, 2, \dots K \,\middle|\, Z_1 = K\right) =$

$\underline{\underline{\text{the } Z_{n-1}^{(k)} \text{ are independent of each other and also of } Z_1}}$

$\mathbb{P}\left(\frac{Z_{n-1}^{(1)}}{\mu^{n-1}} \to 0\right)^K = \theta^K$. The law of total probability gives

$$\theta = \mathbb{P}\left(\frac{Z_n}{\mu^n} \to 0\right) = \sum_{K=0}^{\infty} \mathbb{P}(Z_1 = K)\, \mathbb{P}\left(\frac{Z_n}{\mu^n} \to 0 \,\middle|\, Z_1 = K\right) = \sum_{K=0}^{\infty} p_K \theta^K.$$

That is, $\boxed{\theta = \varphi(\theta)}$

$\boxed{7.3}$ (continued) Since the fixed point equation $\theta = \varphi(\theta)$ has only the two solutions $\theta = 1$ and $\theta = \rho$, we have shown that $\boxed{\text{if } \theta < 1 \text{ then } \theta = \rho}$.

Since for $\mu > 1$ $\left\{ \frac{Z_n}{\mu^n} \to 0 \right\} \supset \left\{ Z_n \to 0 \right\}$, equality of the probabilities also implies that the two events coincide a.s.

$\square$

$\boxed{7.5}$ Z_n is a Galton-Watson branching process with $Z_1 \sim$ Pess Geom(p):

$\mathbb{P}(k \text{ people recruited}) = (1-p)^k p$ for $k = 0, 1, \ldots$. Set $q = 1-p$.

so $\varphi(z) = \sum_{k=0}^{\infty} q^k p z^k = \frac{p}{1-qz}$ and $m = \mathbb{E} Z_1 = \frac{1}{p} - 1$.

$\boxed{\text{I}}$ for $p = \frac{2}{3}$ we have $m = \frac{1}{p} - 1 = \frac{3}{2} - 1 = \frac{1}{2} < 1$. the process is $\boxed{\text{subcritical}}$. So

a.) $\mathbb{P}(\text{extinction}) = \underline{\underline{1}}$

b.) $\mathbb{E} N = \sum_{n=0}^{\infty} \mathbb{E} Z_n = \sum_{n=0}^{\infty} m^n = \frac{1}{1-m} = \underline{\underline{2}}$

c.) irrelevant

$\boxed{\text{II}}$ for $p = \frac{1}{2}$ we have $m = \frac{1}{p} - 1 = 2 - 1 = 1$: the process is $\boxed{\text{critical}}$. So

a.) $\mathbb{P}(\text{extinction}) = \underline{\underline{1}} = 1$

b.) $\mathbb{E} N = \sum_{n=0}^{\infty} \mathbb{E} Z_n = \sum_{n=0}^{\infty} m^n = \sum_{n=0}^{\infty} 1 = \underline{\underline{\infty}}$

c.) irrelevant

$\boxed{\text{III}}$ for $p = \frac{1}{3}$ we have $m = \frac{1}{p} - 1 = 3 - 1 = 2 > 1$: the process is $\boxed{\text{supercritical}}$. So

a.) $\varrho := \mathbb{P}(\text{extinction})$ is the only solution in $[0,1)$ of

$$\varrho = \varphi(\varrho), \text{ so } \varrho = \frac{p}{1-q\varrho} = \frac{1/3}{1-\frac{2}{3}\varrho} = \frac{1}{3-2\varrho}$$

$$\varrho(3-2\varrho) = 1 \Rightarrow 0 = 2\varrho^2 - 3\varrho + 1 = (2\varrho - 1)(\varrho - 1)$$

$$\Downarrow \varrho \neq 1$$

$$\boxed{\varrho = \tfrac{1}{2}}. \quad \boxed{\mathbb{P}(\text{extinction}) = \tfrac{1}{2}}$$

b.) $\mathbb{E} N = \sum_{n=0}^{\infty} 2^n = \underline{\underline{\infty}}$

c.) On the event $\{Z_n \not\to 0\}$ we have $Z_n \sim m^n = 2^n$ in the sense that $\frac{Z_n}{2^n} \to c = c(\omega)$ a.s. on $\{\omega \mid Z_n \not\to 0\}$.

7.6 S is a sum with random number of terms, so

$$\cancel{\mathbb{E}N} \; \mathbb{E}S = \mathbb{E}N \cdot \mathbb{E}(X_i+1) = \frac{1}{p}\left(\frac{1}{p}+1\right) = 6(6+1) = 6 \cdot 7 = \underline{42}.$$

Relation with „Life, the Universe, and Everything":

When Arthur manages to roll a 6, he shouts „I'm almost there".

After the next roll, he will say „Oh, no, I can start all over again" with prob. $\frac{5}{6}$, or „YES, I WON" with prob. $\frac{1}{6}$.

Let N be the number of times he shouts „I'm almost there", and ~~but~~ let X_k be the number of rolls he needs to wait for the first 6 after the previous failure.

So $N \sim \text{PessGeom}\left(\frac{1}{6}\right)$, $X_k \sim \text{PessGeom}\left(\frac{1}{6}\right)$ and

$$S = \sum_{k=1}^{N} (X_k + 1) \text{ is the total number of rolls.}$$

$\boxed{7.9}$ Let $A = \{\text{Aice wins}\}$ and $B = \{\text{Bob wins}\}$.

a.) Let X_n denote the capitel of the casino

• On the event A we have $X_\tau = X_0 + \tau - (16+4)$,

because 1 player goes away with $\cancel{4}$ wins,

and 1 ~ 2 ~ .

• On the event B we have $X_\tau = X_0 + \tau - (0)$,

becaus all ~~to~~ players lost their money.

So $X_\tau = 1\!\!1_A \cdot (X_0 + \tau - 20) + 1\!\!1_B (X_0 + \tau - \cancel{20}^{\,0}) = X_0 + \tau - 20\, 1\!\!1_A$

$\Rightarrow \cancel{\mathbb{E} X_\tau} = \mathbb{E}(X_\tau - X_0) = E\tau - 20\mathbb{P}(A)$

Now the optional stopping theorem says $\mathbb{E}(X_\tau - X_0) = 0$ just like in „ABRACADABRA", so ~~$E\tau - 20\mathbb{P}(A)$~~ =

$$\boxed{E\tau - 20 p_A = 0}$$

b.) Let Y_n denote the capitel of the casino.

• On the event A we have $X_\tau = X_0 + \tau - (8+2)$

. On the event B we have $X_\tau = X_0 + \tau - (16+2)$

so $X_\tau = X_0 + \tau - 10\, 1\!\!1_A - 18\, 1\!\!1_B \Rightarrow \mathbb{E}(X_\tau - X_0) = E\tau - 10 p_A - 18 p_B$

Now the opt. st. thm says $\boxed{E\tau - 10 p_A - 18 p_B = 0}$

c.) $\begin{cases} E\tau - 20 p_A = 0 \\ E\tau - 10 p_A - 18 p_B = 0 \\ p_A + p_B = 1 \end{cases} \Rightarrow \boxed{\begin{array}{l} p_A = \frac{9}{14} \\ p_B = \frac{5}{14} \\ E\tau = \frac{180}{14} \approx 12.9 \end{array}}$