8.3 a.) For any $\{t_1, t_2, \ldots t_n\}$ the random variables $\{X(\frac{1}{t_1}), \ldots X(\frac{1}{t_n})\}$ are jointly Gaussian by assumption. $\{Y(t_1), \ldots, Y(t_n)\}$ are linear „combinations" of these, so they are also jointly Gaussian, thus $Y(t)$ is a Gaussian process. Now we only need to check the covariance structure and continuity.

b.) $EY(t) = t\,\mathbb{E}(X(\frac{1}{t})) = t \cdot 0 = 0$ by assumption ✓ OK

c.) $Cov(Y(t), Y(s)) = Cov(tX(\frac{1}{t}), sX(\frac{1}{s})) = t\,s\,Cov(X(\frac{1}{t}), X(\frac{1}{s})) =$

$$\underset{\text{Wiener}}{\overset{X(t)\text{ is}}{=}} t\,s \min\{\tfrac{1}{t}, \tfrac{1}{s}\} \underset{t,s>0}{=} \min\{\tfrac{ts}{t}, \tfrac{ts}{s}\} = \min\{t, s\}$$ ✓ OK

d.) For those ω where $t \to X(t)$ is continuous (which is $\mathbb{P}$-a.e. ω), $t \mapsto tX(\frac{1}{t}) = Y(t)$ is also continuous, at least for $t \in (0, \infty)$.

[In $t = 0$ continuity follows from the covariance structure:
as $s \to \infty$, $Y(\frac{1}{s}) = \frac{1}{s}X(s) \to 0$ a.s.
→ in step 1, on $s \in \mathbb{Q}$ (by the law of large numbers)
→ then by continuity of $X(s)$ also a.s. for all s]

8.4

$\mathbb{P}(|\xi_n| > 2\ln n) \leq \sqrt{\frac{2}{\pi}} e^{-\frac{(2\ln n)^2}{2}} = \sqrt{\frac{2}{\pi}} e^{-2\ln^2 n}$ by the inequality given.

Now for $n \geq 3$ we have $\ln n > 1$, so $\ln^2 n > \ln n$,

so $-2\ln^2 n < -2\ln n$, so $e^{-2\ln^2 n} < e^{-2\ln n} = \frac{1}{n^2}$,

That is, $\mathbb{P}(|\xi_n| > 2\ln n) \leq \sqrt{\frac{2}{\pi}} \frac{1}{n^2}$ for $n > 3$.

This implies that $\sum_{n=1}^{\infty} \mathbb{P}(|\xi_n| > 2\ln n) < \infty$.

The Borel-Cantelli lemma now gives the statement.

(Independence of the ξ_n is not needed.)

8.5 The construction uses the i.i.d. standard Gaussian random variables $\xi_{1,1}$ $\rightsquigarrow$ used in g_1

$\xi_{2,1}, \xi_{2,2}$ $\to$ used in g_2

$\xi_{3,1}\ \xi_{3,2}, \xi_{3,3}, \xi_{3,4}, \ldots$ $\to$ used in g_3

Let's write thes into a single sequence, so

g_1 uses ξ_1

g_2 uses ξ_2, ξ_3

g_3 uses $\xi_4, \xi_5, \xi_6, \xi_7$

$\vdots$

g_m uses $\xi_{2^{m-1}}, \xi_{2^{m-1}+1}, \ldots, \xi_{2^m-1}$.

With this notation $\sup_{x\in[0,1]} |g_m(x)| = \max_{2^{m-1}\le n<2^m} \frac{1}{2^m}|\xi_n|$.

With a little generousity we write $\sup|g_m| \le \frac{1}{2^m}\max_{n<2^m}|\xi_n|$.

Now restrict our attention to the event

$A := \{ |\xi_n| \le 2\ln n$ for every n, except for at most finitely many n-s$\}$.

By HW 8.4, we have $\mathbb{P}(A)=1$. On this event A

$$\sup|g_m| \le \frac{1}{2^m}\max_{n<2^m} 2\ln n = \frac{1}{2^m} 2\ln(2^m) = \frac{1}{2^m} 2m\ln m =: C_m$$

except for at most finitely many m-s.

Since $\sum_m C_m < \infty$, we get that on the event A the series $\sum_{m=0}^{\infty} g_m$ is uniformly absolutely convergent. $\square$