[9.1] (Durrett 8.1.3) For fixed n, the

a.) $\Delta_{m,n} = B(tm2^{-n}) - B(t(m-1)2^{-n})$ are independent for $m = 1, 2, \dots, 2^n$, and $\Delta_{m,n} \sim N(0; t2^{-n})$, which also implies that $\mathbb{E}\Delta_{m,n} = 0$ and

$$\mathbb{E}(\Delta_{m,n}^2) = Var\,\Delta_{m,n} = t2^{-n}$$

So let $X_{m,n} = \Delta_{m,n}^2 - t2^{-n}$, ~~and introduce~~ we have

$$\mathbb{E}\left(\left(\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2\right) - t\right)^2\right) = \mathbb{E}\left[\left(\sum_{m=1}^{2^n} X_{m,n}\right)^2\right] \overset{\text{independence}}{\underset{\mathbb{E}X_{m,n}=0}{=}} 2^n\,Var\,X_{1,n}$$

But ~~$Var X_{1,n}$~~ let $\Delta_{1,n} = 2^{-n/2}\sqrt{t}\,Y$ where $\boxed{Y \sim N(0,1)}$ so

$$\Delta_{1,n}^2 = 2^{-n}tY^2, \quad X_{1,n} = t2^{-n}(Y^2-1) \text{ and}$$

$$X_{1,n}^2 = t^2 2^{-2n}(Y^2-1)^2, \text{ so}$$

$$Var\,X_{1,n} = \mathbb{E}X_{1,n}^2 = t^2 2^{-2n}\,\mathbb{E}((Y^2-1)^2)$$

Define $\boxed{c := \mathbb{E}((Y^2-1)^2)}$ Clearly $c < \infty$, because all moments of Y are finite.

$$\text{So } \mathbb{E}\left[\left(\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2\right) - t\right)^2\right] = 2^n t^2 2^{-2n} c = \frac{ct^2}{2^n}$$

The Markov inequality and the Borel Cantelli lemma imply that for every $\varepsilon > 0$, $\mathbb{P}\left(\left|\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right| > \varepsilon\right) \leq \frac{ct^2}{\varepsilon^2 2^n}$, which is convergent,

so $\left|\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right| > \varepsilon$ for at most finitely many n-s almost surely for every $\varepsilon \in \{1; \frac{1}{2}; \frac{1}{3}; \frac{1}{4}, \dots\}$.

$\Rightarrow \sum_{m=1}^{2^n} \Delta_{m,n}^2 \to t$ a.s. $\square$

$\boxed{9.2}$ (Durrett 8.2.3)

a.) Since the previous theorem claims that almost surely B_t returns to 0 arbitrarily soon, and $B_{t+a} - B_a$ is also a Wiener process, almost surely $\exists\ \xi \in (a,b)$ such that $B_\xi(\omega) = B_a(\omega)$ Since $B(\omega)$ is continuous, ~~there has~~ ~~to~~ it obtains its maximum on $[a,\xi]$ at some $\xi' \in [a,\xi] \subset [a,b]$. This ξ' will do for a local maximum-place unless $\xi' = a$ or $\xi' = \xi$,
but in this case we also find a $\xi'' \in (a,\xi)$ with $B_{\xi''} = 0$, and this $\xi' \in (a,b)$ is a local max.-place. $\square$

b.) $\mathbb{P}(\exists\ a<b$ s.t. $a,b \in \mathbb{Q}$ and there's no local max. in $(a,b)) \leq$

$$\leq \sum_{\substack{a<b \\ a,b\in\mathbb{Q}}} \mathbb{P}(\text{no local max. in } (a,b)) = \sum_{\substack{\text{countable} \\ \text{set}}} 0 = 0 \quad \square$$