Stochastic Analysis
Problem Set 4
Stochastic differential equations

4.1 Check that the following processes solve the corresponding SDE's, where $B(t)$ is 1-dimensional standard Brownian motion:

(a) $X(t) = e^{B(t)}$, with $B(0) = b$ solves

$$dX(t) = \frac{1}{2}X(t)dt + X(t)dB(t), \qquad X(0) = e^b.$$

(b) $X(t) = \frac{B(t)}{1+t}$, with $B_0 = b$, solves

$$dX(t) = -\frac{X(t)}{1+t}dt + \frac{1}{1+t}dB_t, \qquad X(0) = b.$$

(c) $X(t) = \sin B(t)$, with $B(0) = b \in (-\pi/2, \pi/2)$, and $t < \min\{t : |B(t)| = \pi/2\}$, solves

$$dX(t) = -\frac{1}{2}X(t)dt + \sqrt{1 - X(t)^2}dB_t, \qquad X(0) = \sin b.$$

(d) $\big(X_1(t), X_2(t)\big) = \big(\cosh B(t), \sinh B(t)\big)$, with $B(0) - b$, solves

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dt + \begin{pmatrix} X_2(t) \\ X_1(t) \end{pmatrix} dB(t).$$

4.2 Let $B(t)$ be a standard 1-dimensional Brownian motion with $B(0) = b$, and $\big(U(t), V(t)\big) := \big(\cos B(t), \sin B(t)\big)$. Write down in vectorial notation the SDE driving the 2-dimensional process $\big(U(t), V(t)$.

4.3 Solve the following SDE's, where $B(t)$ is 1-dimensional standard Brownian motion starting from $B(0) = 0$:

(a)

$$dX(t) = -X(t)dt + e^{-t}dB(t).$$

(b)

$$dX(t) = rdt + \alpha X(t)dB(t),$$

with $r, \alpha \in \mathbb{R}$ constants.
Hint: Multiply by $\exp\big(-\alpha B(t) + \frac{\alpha^2}{2}t\big)$.

(c) Now, $X(t) = \big(X_1(t), X_2(t)\big) \in \mathbb{R}^2$, and $B(t) = \big(B_1(t), B_2(t)\big)$ is standard 2-dimensional Brownian motion.

$$\begin{aligned} dX_1(t) &= \quad X_2(t)dt + \alpha dB_1(t) \\ dX_2(t) &= -X_1(t)dt + \beta dB_2(t), \end{aligned}$$

or in vector notation,

$$dX(t) = JX(t)dt + AdB(t), \qquad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Hint: Multiply by left by e^{-Jt}.

4.4 The Ornstein-Uhlenbeck process:

(a) Solve explicitly the stochastic differential equation

$$dX(t) = -\gamma X(t)dt + adB(t), \qquad X(0) = x_0,$$

and show that the process $X(t)$ is Gaussian.
Hint: Multiply by $e^{\gamma t}$.

(b) Compute $\mathbf{E}(X(t))$ and $\mathbf{Cov}\big(X(s), X(t)\big)$.

(c) **Bonus exercise:** Let $Y_k^{(n)}$ be the Markov chain on the state space $S^{(n)} := \{0, 1, \ldots, n\}$ with transition matrix

$$P_{i,j}^{(n)} = \frac{i}{n}\delta_{i-1,j} + \frac{n-i}{n}\delta_{i+1,j}, \qquad i, j \in S^{(n)}.$$

The Markov chain $Y_k^{(n)}$ is called *Ehrenfest's Urn Model* (or *Dogs and Fleas*). Define the sequence of continuous time processes

$$X^{(n)}(t) := \frac{Y_{\lfloor nt \rfloor}^{(n)} - (n/2)}{\sqrt{n}}, \qquad t \geq 0.$$

Write down an *approximate stochastic differential equation* for $X^{(n)}(t)$, with time increments $dt = \frac{1}{n}$ and conclude (non-rigorously) that the Ornstein-Uhlenbeck process is – in some sense – the limit of the processes $X^{(n)}(t)$ (that is: the *scaling limit* of Ehrenfest's Urn Model.)

4.5 Write down the infinitesimal generator as elliptic differential operator for the following Itô diffusions:

(a) $dX(t) = \beta dt + \alpha X(t) dB(t)$.

(b) $dY(t) = \begin{pmatrix} dt \\ dX(t) \end{pmatrix}$, where $dX(t) = -\gamma X(t) dt + \alpha dB(t)$.

(c) $\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ X_2(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ e^{X_1(t)} \end{pmatrix} dB(t)$.

(d) $\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & X_1 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}$.

4.6 Find an Itô diffusion (i.e., write down the SDE for it) whose infinitesimal generator is the following:

(a) $Af(x) = f'(x) + f''(x)$, $f \in C_0^2(\mathbb{R})$.

(b) $Af(t,x) = \dfrac{\partial f}{\partial t} + cx\dfrac{\partial f}{\partial x} + \dfrac{1}{2}\alpha^2 x^2 \dfrac{\partial^2 f}{\partial x^2}$, $f \in C_0^2(\mathbb{R}^2)$.

4.7 Let $X(t)$ be a geometric Brownian motion, i.e. strong solution of the following SDE

$$dX(t) = \beta X(t) dt + \alpha X(t) dB(t), \qquad X_0 = x > 0,$$

where $\alpha > 0$, $\beta \in \mathbb{R}$ are fixed parameters.

(a) Find the generator A of the diffusion $t \mapsto X(t)$ and compute $Af(x)$ when $f : \mathbb{R}_+ \to \mathbb{R}$ is $f(x) = x^\gamma$, γ constant.

(b) Let $0 < r < R < \infty$, and $r \le x \le R$. using Dynkin's formula, compute

$$\mathbf{P}\big(\tau_r < \tau_R \mid X(0) = x\big),$$

where τ_r, and τ_R are the first hitting times of r, respectively, R.
Hint: Solve the boundary value problem $Af(x) = 0$ for $r < x < R$, with $f(r) = 1$, $f(R) = 0$.

(c) Assume $\beta < \alpha^2/2$. What is $\mathbf{P}\big(X(t)$ ever hits $R \mid X(0) = x\big)$?

(d) Assume $\beta > \alpha^2/2$. What is $\mathbf{P}\big(X(t)$ ever hits $r \mid X(0) = x\big)$?

4.8 (a) Find the generator of the δ-dimensional Bessel process, $BES(\delta)$

$$dY^{(\delta)}(t) = \frac{\delta - 1}{2Y^{(\delta)}(t)} dt + dB(t)$$

on $\mathbb{R}_+$.

(b) Let $0 < r < R < \infty$, and $r \leq x \leq R$. using Dynkin's formula, compute

$$\mathbf{P}\big(\tau_r < \tau_R \bigm| Y^{(\delta)}(0) = x\big),$$

where τ_r, and τ_R are the first hitting times of r, respectively, R.
Hint: Solve the boundary value problem $Af(x) = 0$ for $r < x < R$, with $f(r) = 1$, $f(R) = 0$. Note that the solutions are *qualitatively different* for $\delta \in [0, 2)$, $\delta = 2$, respectively, $\delta > 2$.

(c) Show that $BES(\delta)$ is transient if $\delta > 2$.

(d) Show that $BES(2)$ almost surely hits all points in $(0, \infty)$, but never hits 0.

(e) Show that for $\delta \in [0, 2)$ $BES(\delta)$ almost surely hits 0 (no matter where it starts).