Stochastic Processes, Problem Set 2: Solutions

2.1 (b) $\Rightarrow$ (a): evident

(a) $\Rightarrow$ (b): do it for $\boxed{n=2}$

$$\mathbb{E}\left(F(X(t+u_1), X(t+u_2)) \mid \mathcal{F}_t^X\right) =$$

$$\mathbb{E}\left(\mathbb{E}\left(F(X(t+u_1), X(t+u_2)) \mid \mathcal{F}_{t+u_1}^X\right) \mid \mathcal{F}_t^X\right) =$$

$$\mathbb{E}\Big(\underbrace{\mathbb{E}\left(F(X(t+u_1), X(t+u_2)) \mid \sigma(X(t+u_1))\right)}_{G(X(t+u_1))} \mid \mathcal{F}_t^X\Big) =$$

$$\mathbb{E}\left(\mathbb{E}\left(F(X(t+u_1), X(t+u_2)) \mid \sigma(X(t+u_1))\right) \mid \sigma(X(t))\right) =$$

$$\mathbb{E}\left(\mathbb{E}\left(F(X(t+u_1), X(t+u_2)) \mid \sigma(X(t+u_1), X(t))\right) \mid \sigma(X(t))\right) =$$

$$\mathbb{E}\left(F(X(t+u_1), X(t+u_2)) \mid \sigma(X(t))\right)$$

$\square$ ✓

2.2

$$\boxed{0 < s < t}$$

(a) $\mathbb{E}\left(B(t)^2 \mid \mathcal{F}_s^B\right) =$

$\mathbb{E}\left(B(s)^2 + 2\left(B(t) - B(s)\right)B(s) + \left(B(t) - B(s)\right)^2 \mid \mathcal{F}_s^B\right)$

$= \cdots = B(s)^2 + \mathbb{E}\left(\left(B(t) - B(s)\right)^2 \mid \mathcal{F}_s^B\right) \geq B(s)^2$

(b) $\mathbb{E}\left(Y(t) \mid \mathcal{F}_s\right) = \mathbb{E}\left(\Psi(X(t)) \mid \mathcal{F}_s\right) \overset{\text{Jensen's ineq}}{\geq}$

$\Psi\left(\mathbb{E}\left(X(t) \mid \mathcal{F}_s\right)\right) = \Psi\left(X(s)\right) = Y(s)$

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2.3 $\mathbb{E}\left(B(t) \mid \mathcal{F}_s^B\right) =$

$\mathbb{E}\left(B(s) \mid \mathcal{F}_s^B\right) + \underbrace{\mathbb{E}\left(B(t) - B(s) \mid \mathcal{F}_s^B\right)}_{= 0 \checkmark} =$

$B(s)$

Let $X(t) := B(t)^2 - t$

$$E\left(X(t) \mid \mathcal{F}_s^B\right) = \dots$$

$$E\left(X(s) - (t-s) + (B(t)-B(s))^2 + 2(B(t)-B(s))B(s) \mid \mathcal{F}_s\right)$$

$$= \dots = X(s)$$

Let $X(t) := B(t)^3 - 3tB(t)$

then

$$X(t) - X(s) = (B(t)-B(s))^3 + 3(B(t)-B(s))^2 B(s) + 3(B(t)-B(s))B(s)^3 - 3t(B(t)-B(s)) - 3(t-s)B(s)$$

Hence

$$E\left(X(t) - X(s) \mid \mathcal{F}_s^B\right) = 0$$

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2.4 (3b)

(a) $E\left(B(t)+t \,\middle|\, \mathcal{F}_s^B\right) = B(s)+t$

$\neq B(s)+s$

not martingale

(b) $E\left(B(t)^2 \,\middle|\, \mathcal{F}_s^B\right) = B(s)^2 + (t-s)$

$\neq B(s)^2$

not martingale

(c) $E\left(t^2 B(t) - 2\int_0^t r\,B(r)\,dr \,\middle|\, \mathcal{F}_s^B\right) =$

$t^2 B(s) - 2\int_0^s r\,B(r)\,dr - 2\int_s^t r\,B(s)\,dr$

$= s^2 B(s) - 2\int_0^s r\,B(r)\,dr$

this one is a martingale

(d) $E\left(B_1(t)B_2(t) \,\middle|\, \mathcal{F}_s^B\right) = B_1(s)B_2(s)$

it is a martingale

(4)

2.5 (a) The OST can be applied, since

$$\sup_s |B^{\tau}(s)| \leq \max\{a, b\} < \infty.$$

$$0 = \mathbb{E}\left(B(\tau)\right) =$$

$$= -a\, \mathbb{P}(\tau_l < \tau_r) + b\, \mathbb{P}(\tau_l > \tau_r)$$

$$\mathbb{P}(\tau_l < \tau_r) + \mathbb{P}(\tau_l > \tau_r) = 1$$

Hence $\mathbb{P}(\tau_l < \tau_r) = \dfrac{b}{a+b}$

Apply the OST to $B(t)^2 - t$,

$$0 = \mathbb{E}\left(B(\tau)^2 - \tau\right)$$

$$E(\tau) = \mathbb{E}\left(B(\tau)^2\right) = a^2 \cdot \frac{b}{a+b} + b^2 \frac{a}{a+b}$$

$$= a \cdot b.$$

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25 (a)

$$\mathbb{E}\left(e^{\theta B(t) - \theta^2 t/2} \,\middle|\, \mathcal{F}_s^B\right) =$$

$$\mathbb{E}\left(e^{\theta B(s) - \frac{\theta^2 s}{2}} \cdot e^{\theta(B(t)-B(s)) - \frac{\theta^2(t-s)}{2}} \,\middle|\, \mathcal{F}_s^B\right) =$$

$$e^{\theta B(s) - \frac{\theta^2 s}{2}} \underbrace{\mathbb{E}\left(e^{\theta(B(t)-B(s)) - \frac{\theta^2(t-s)}{2}} \,\middle|\, \mathcal{F}_s^B\right)}_{=1}$$

$$= e^{\theta B(s) - \frac{\theta^2 s}{2}}$$

(b) $$\frac{d}{d\theta}\left(e^{\theta B - \frac{\theta^2 s}{2}}\right) = (B - \theta s)\, e^{\theta B - \frac{\theta^2 s}{2}}$$

$$\frac{d^2}{d\theta^2}\left(\text{---}\right) = \left((B-\theta s)^2 - s\right) e^{\theta B - \frac{\theta^2 s}{2}}$$

$$\frac{d^3}{d\theta^3}\left(-\right) = \left((B-\theta s)^3 - 3s(B-\theta s)\right)e^{\theta B - \frac{\theta^2}{2}s}$$ 6

$$\frac{d^4}{d\theta^4}\left(-\right) = \left((B-\theta s)^4 - 6s(B-\theta s)^2 + 3s^2\right)e^{\theta B - \frac{\theta^2}{2}s}$$

$$\frac{d^4}{d\theta^4}\left(e^{\theta B - \frac{\theta^2}{2}s}\right)\Bigg|_{\theta=0} = B^4 - 6sB^2 + 3s^2$$

$$t \longmapsto B(t)^4 - 6t\,B(t)^2 + 3t^2$$

is a martingale

c) $$\frac{d^n}{d\theta^n} e^{\theta B(t) - \frac{\theta^2}{2}t}\Bigg|_{\theta=0} = t^{n/2} H_n\left(\frac{B(t)}{\sqrt{t}}\right).$$

$$e^{\theta B(t) - \frac{\theta^2 t}{2}} = e^{\theta\sqrt{t}\,\frac{B(t)}{\sqrt{t}} - \frac{(\theta\sqrt{t})^2}{2}}$$

Let $\lambda := \theta\sqrt{t}$, $\frac{d^n}{d\theta^n} = t^{n/2}\frac{d^n}{d\lambda^n}$

$$\frac{d^n}{d\theta^n} e^{\theta B(t) - \frac{\theta^2 t}{2}} = t^{n/2}\frac{d^n}{d\lambda^n} e^{-\frac{1}{2}\left(\lambda - \frac{B(t)}{\sqrt{t}}\right)^2 + \frac{B(t)^2}{2t}}$$

$$= t^{n/2} e^{+\frac{B(t)^2}{2t}} \frac{d^n}{d\lambda^n} e^{-\frac{1}{2}\left(\lambda - \frac{B(t)}{\sqrt{t}}\right)^2}$$

$$= t^{n/2} e^{\frac{B(t)^2}{2t}} e^{-\frac{1}{2}\left(\lambda - \frac{B(t)}{\sqrt{t}}\right)^2} H_n\left(\lambda - \frac{B(t)}{\sqrt{t}}\right)$$

now let $\theta = 0$ (that is : $\lambda = 0$)

and get

$$\left.\frac{d^n}{d\theta^n} e^{\theta B(t) - \frac{\theta^2 t}{2}}\right|_{\theta=0} = t^{n/2} H_n\left(\frac{B(t)}{\sqrt{t}}\right) \quad \Box.$$

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2.7 $\mathbb{E}\left(e^{\theta B(\tau) - \frac{\theta^2}{2}\tau}\right) = 1$ $\boxed{\tau \text{ and } B(\tau) \text{ are } \boxed{\text{independent}}}^{-1}$

$$\mathbb{E}\left(e^{-\lambda\tau}\right) = \left(\mathbb{E}\left(e^{\sqrt{2\lambda}\,B(\tau)}\right)\right)^{-1}$$

$$= \left(\cosh\left(\sqrt{2\lambda}\right)\right)^{-1}$$

2.8 Similarly:

$$\mathbb{E}\left(e^{-\lambda\tau}\right) = \left(\mathbb{E}\left(e^{\sqrt{2\lambda}\,B_1(\tau)}\right)\right)^{-1}$$

$$\mathbb{E}\left(e^{\sqrt{2\lambda}\,B_1(\tau)}\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{2\lambda}\cos\varphi}\,d\varphi$$

since $(B_1(\tau), B_2(\tau))$ is uniformly distributed on S^1