Stochastic Analysis
Problem Set 2, some solutions
Filtrations, Stopping Times, Markov Property, Martingales, ...

2.9 Let $B(t)$ be a standard Brownian motion and let ξ be a random variable with *Bernoulli* $\left(\frac{1}{2}\right)$ distribution, independent of $B(t)$. Let $X(t) = \xi(1 + B(t))$. Show that $X(t)$ is Markov but not strongly Markov (w.r.t. the natural filtration).

Solution: First the interpretation: we toss a fair coin. If the result is "tails" (denoted as $\xi = 0$), then $X(t)$ is constant 0. If the result is "heads" (denoted as $\xi = 1$), then $X(t)$ is a Brownian motion starting from 1.

a.) This $X(t)$ is clearly not strongly Markov: if $\tau := \inf\{t \geq 1 \,|\, X(t) = 0\}$ is the first hitting time of 0 after time 1, then $X(\tau) = 0$ deterministically, so $\sigma(X(\tau))$ is the trivial (indiscrete) σ-algebra, containing no information, so

$$\mathbf{E}\big(F(X(\tau + u)) \;\big|\; \sigma(X(\tau))\big) = \mathbf{E}\left(F(X(\tau + u))\right)$$

is a constant for any bounded and measurable $F : \mathbb{R} \to \mathbb{R}$. As an example, let $F : \mathbb{R} \to \mathbb{R}$ be the indicator function of 0. Then, for any $u > 0$

$$\mathbf{E}\big(F(X(\tau + u)) \;\big|\; \sigma(X(\tau))\big) = \mathbf{E}\left(F(X(\tau + u))\right) = \mathbf{P}\left(X(\tau + u) = 0\right) = \frac{1}{2}$$

(since $\mathbf{P}\big(X(\tau + u) = 0 \;\big|\; \xi = 0\big) = 1$ and $\mathbf{P}\big(X(\tau + u) = 0 \;\big|\; \xi = 1\big) = 0$). On the other hand $\mathcal{F}_\tau$ is not trivial: it definitely contains the events $\{\xi = 0\}$ and $\{\xi = 1\}$. (If you see the trajectory up to τ, you can tell the result of the coin toss.) So $\mathbf{E}\big(F(X(\tau + u)) \;\big|\; \mathcal{F}_\tau\big)$ is *not* constant (in particular it is 1 on $\{\xi = 0\}$ and 0 on $\{\xi = 1\}$).

In summary: for this particular τ and F, and for any $u > 0$

$$\mathbf{E}\big(F(X(\tau + u)) \;\big|\; \mathcal{F}_\tau\big) \neq \mathbf{E}\big(F(X(\tau + u)) \;\big|\; \sigma(X(\tau))\big),$$

so the process is not strongly Markov.

b.) The surprising part is that $X(t)$ is Markov. This is because, for every fixed *deterministic* $t \in \mathbb{R}$, $\mathbf{P}(1 + B(t) = 0) = 0$. This way if we see that $X(t) = 0$, then we know that $\xi = 0$ (or a zero probability event has occured). So

$$X(t+u) = \begin{cases} 0 & \text{almost surely on } \{X(t) = 0\} \\ 1 + B(t+u) & \text{surely on } \{X(t) \neq 0\}. \end{cases}$$

This means

$$\begin{aligned} \mathbf{E}\big(F(X(t+u)) \mid X(t)\big) &= \begin{cases} F(0) & \text{a.s. on } \{X(t) = 0\} \\ \mathbf{E}\big(F(1 + B(t+u)) \mid B(t)\big) & \text{a.s. on } \{X(t) \neq 0\}. \end{cases} \\ &= \begin{cases} F(0) & \text{a.s. on } \{\xi = 0\} \\ \mathbf{E}\big(F(1 + B(t+u)) \mid B(t)\big) & \text{a.s. on } \{\xi = 1\}. \end{cases} \end{aligned}$$

We need to see that $\mathbf{E}\big(F(X(t+u)) \mid \mathcal{F}_t\big)$ is the same. This can be seen by using that

- $\mathcal{F}_t = \sigma(\mathcal{F}_t^B, \xi)$,
- $X(t+u) = \xi(1 + B(t+u))$ where ξ is $\mathcal{F}_t$-measurable,
- $B(t+u)$ is independent of ξ,
- and $B(t)$ is Markov.

2.10 a.) Show that if $X(t)$ is a submartingale, $\psi : \mathbb{R} \to \mathbb{R}$ is convex and *increasing* such that $\mathbf{E}(|\psi(X(t)|) < \infty$ for every t, then $Y(t) := \psi(X(t))$ is also a submartingale.

b.) Give an example of a submartingale $X(t)$ such that $Y(t) := (X(t))^2$ is not a submartingale.

Solution:

a.)
i.) Adaptedness is understood w.r.t the natural filtration, so it is automatic.
ii.) Integrability is assumed explicitly as $\mathbf{E}(|Y(t)|) = \mathbf{E}(|\psi(X(t))|) < \infty$.
iii.) The essence is the submartingale property: for $t, u \geq 0$

$$\mathbf{E}\big(Y(t+u) \mid \mathcal{F}_t\big) = \mathbf{E}\big(\psi(X(t)) \mid \mathcal{F}_t\big) \overset{(1)}{\geq} \psi\left(\mathbf{E}\big(X(t) \mid \mathcal{F}_t\big)\right) \overset{(2)}{\geq} \psi(X(t)) = Y(t).$$

In step (1) we used Jensen's inequality. In step (2) we used that $\mathbf{E}\big(X(t) \mid \mathcal{F}_t\big) \geq X(t)$ by the submartingale property of $X(t)$ *and the monotonicity of* ψ.

b.) Note that $t \in [0, \infty)$. The increasing deterministic function $X(t) = -\frac{1}{t}$ does the job, since it's a submartingle, but $Y(t) := (X(t))^2 = \frac{1}{t^2}$ is (strictly) decreasing, so it's a supermartingale (and not a submartingale).

2.11 Let $B(t)$ be a standard Brownian motion and let $X(t) = B(t) - \frac{t}{2}$: a kind of "Brownian motion with drift to the left". Let $-a < 0 < b$, let $\tau_{left} = \inf\{t \in \mathbb{R}^+ \mid X(t) = -a\}$ and $\tau_{right} = \inf\{t \in \mathbb{R}^+ \mid X(t) = b\}$ be the first hitting times for $-a$ and b, and let $\tau = \inf\{\tau_{left}, \tau_{right}\}$. Let $p_{left} = p_{left}(a, b) = \mathbf{P}(\tau_{left} < \tau_{right})$ be the probability that $-a$ is reached sooner than b, and $p_{right} = p_{right}(a, b) = \mathbf{P}(\tau_{right} < \tau_{left})$ be the probability that b is reached sooner than $-a$.

a.) Show that $p_{left} + p_{right} = 1$, which means exactly that either $-a$ or b is almost surely reached. (This is the same as saying that $\tau < \infty$ almost surely.)

b.) Find a number $q > 0$ such that $M(t) := q^{X(t)}$ is a martingale.

c.) Apply the optional stopping theorem to $M(t)$ and τ to find p_{left} and p_{right}.

d.) Find the probability that $X(t)$ ever reaches $+1$. *(Hint: set $b = 1$, and look at $\lim p_{right}(a, b)$ as $a \to \infty$.)*

Solution:

a.) Clearly $\tau \le \tau_{left}$, and I claim that $\tau_{left} < \infty$ almost surely. Indeed, for big t the particle is very likely to be left of $-a$, because the expected position is $-\frac{t}{2}$, while the fluctuation around that is only around $\sqrt{t}$. With a rigorous calculation:

$$\mathbf{P}(X(t) > -a) = \mathbf{P}\left(\mathcal{N}\left(-\frac{t}{2}, t\right) > -a\right) = 1 - \Phi\left(\frac{-a - (-\frac{t}{2})}{\sqrt{t}}\right) =$$
$$= 1 - \Phi\left(\frac{\sqrt{t}}{2} - \frac{a}{\sqrt{t}}\right) \to 0,$$

so

$$\mathbf{P}(X(t) > -a \text{ for every } t > 0) = 0.$$

b.) We see from Exercise 2.6 (with $\theta = 1$) that $q = e$ does the job: $M(t) := e^{X(t)} = \exp\{B(t) - \frac{t}{2}\}$ is a martingale.

c.) For $t \le \tau$ we have $B(t) \le b$, so $B(t \wedge \tau) \le b$, implying that $0 < M(t \wedge \tau) \le e^b$, so the stopped martingale is bounded. In part a.) we have seen that $\tau < \infty$ almost surely, so the optional stoping theorem can be applied, and it gives that $\mathbf{E}(M(\tau)) = M(0) = 1$. But $X(\tau) = -a$ on $\{\tau_{left} < \tau_{right}\}$ and $X(\tau) = b$ on

$\{\tau_{right} < \tau_{left}\}$, so $1 = \mathbf{E}(M(\tau)) = \mathbf{E}\left(e^{X(\tau)}\right) = p_{left}e^{-a} + p_{right}e^{b}$. Together with part a.) we have the system of equations

$$\begin{cases} p_{left} + & p_{right} = 1 \\ e^{-a}p_{left} + e^{b} & p_{right} = 1 \end{cases}.$$

The unique solution is

$$p_{left} = \frac{e^{b} - 1}{e^{b} - e^{-a}}$$
$$p_{right} = \frac{1 - e^{-a}}{e^{b} - e^{-a}}.$$

d.) A fixed $b := 1 > 0$ is reached if and only if b is reached sooner than $-a$ for some $a > 0$. So

$$\mathbf{P}(\{b = 1 \text{ is reached}\}) = \lim_{a\to\infty} p_{right}(a, b = 1) = \lim_{a\to\infty} \frac{1 - e^{-a}}{e - e^{-a}} = \frac{1}{e}.$$