4.1 Straightforward applications of Itô's formula.

4.2 $\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = \begin{pmatrix} \cos B(t) \\ \sin B(t) \end{pmatrix}$

by applying Itô's formula one gets

$$d\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} dt + \begin{pmatrix} -V(t) \\ U(t) \end{pmatrix} dB(t).$$

4.4

a) $$X(t) = X(0)e^{-\gamma t} + a\int_0^t e^{-\gamma(t-s)}\,dB(s)$$

b) $$\mathbb{E}(X(t)) = x_0 e^{-\gamma t}$$

let $0 \leq s \leq t < \infty$:

$$\mathrm{Cov}(X(t), X(s)) =$$

$$a^2\mathbb{E}\left(\int_0^t e^{-\gamma(t-u)}\,dB(u) \cdot \int_0^t e^{-\gamma(s-v)}\,dB(v)\right) =$$

$$a^2\int_0^s e^{-\gamma(t+s-2u)}\,du =$$

$$\frac{a^2}{2\gamma}\left(e^{-\gamma|t-s|} - e^{-\gamma(t+s)}\right)$$

(c) $$\mathbb{E}\left(Y^{(n)}_{k+1} - Y^{(n)}_{k} \,\middle|\, Y^{(n)}_{k} = y\right) = -\frac{2}{n}\left(y - \frac{n}{2}\right)$$

$$\mathrm{Var}\left(Y^{(n)}_{k+1} - Y^{n}_{k} \,\middle|\, Y^{(n)}_{k} = y\right) = 1 - \frac{4}{n^2}\left(y - \frac{n}{2}\right)^2$$

May think abou [illegible] us:

$$Y^{(n)}_{k+1} - Y^{(n)}_{k} = -\frac{2}{n}\left(Y^{(n)}_{k} - \frac{n}{2}\right) + \xi^{(n)}_{k+1} + \text{small error}\ \cancel{\xi^{(n)}_{k+1}}$$

random variable independent of the past

$$\mathbb{E}\left(\xi^{(n)}_{k+1}\right) = 0$$

$$\mathrm{Var}\left(\xi^{(n)}_{k+1}\right) = 1$$

Now, let $X^{(n)}(t) := \frac{1}{\sqrt{n}}\left(Y^{(n)}_{\lfloor nt \rfloor} - \frac{n}{2}\right)$

$$dt \simeq \frac{1}{N}$$

to get

$$dX^{(n)}(t) \approx -2X^{(n)}(t)\, dt + dB(t) \ldots$$

4.5

(a) $Af(x) = \beta f'(x) + \frac{\alpha^2}{2} x^2 f''(x)$

(b) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $\quad f = f(u, x)$

$$Af(u,x) = \frac{\partial f}{\partial u}(u,x) - \gamma x \frac{\partial f}{\partial x}(u,x) + \frac{\alpha^2}{2} \frac{\partial^2 f}{\partial x^2}(u,x)$$

(c) $f : \mathbb{R}^2 \to \mathbb{R}$

$$Af(x_1,x_2) = \frac{\partial f}{\partial x_1}(x_1,x_2) + x_2 \frac{\partial f}{\partial x_2}(x_1,x_2) + \frac{1}{2} e^{2x_1} \frac{\partial^2 f}{\partial x_2^2}(x_1,x_2)$$

(d) $f : \mathbb{R}^2 \to \mathbb{R}$, $\quad f = f(x_1, x_2)$

$$Af(x_1,x_2) = \frac{\partial f}{\partial x_1}(x_1,x_2) + \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2}(x_1,x_2) + \frac{\alpha^2}{2} \frac{\partial^2 f}{\partial x_2^2}(x_1,x_2)$$

4.6

(a) $dX(t) = dt + \sqrt{2}\, dB(t)$

(b) $$dY(t) = \begin{pmatrix} dt \\ cX(t)\,dt + \alpha X(t)\,dB(t) \end{pmatrix}$$

4.8

$$\boxed{x > 0}$$

$$Af(x) = \frac{\delta - 1}{2x} f'(x) + \frac{1}{2} f''(x)$$

$Af(x) = 0$ general sln:

$$f(x) = \begin{cases} C_1 + C_2\, x^{2-\delta} & \delta \neq 2 \\ C_1 + C_2 \ln x & \delta = 2 \end{cases}$$

b) $0 < r < x < R < \infty$

$$\mathbb{P}_x(\tau_r < \tau_R) =$$

$$\begin{cases} \dfrac{x^{2-\delta} - R^{2-\delta}}{r^{2-\delta} - R^{2-\delta}} & \text{if } \delta \neq 2 \\[2ex] \dfrac{\ln x - \ln R}{\ln r - \ln R} & \text{if } \delta = 2 \end{cases}$$

(c) $\boxed{\delta > 2}$

$$\lim_{R\to\infty} P(\tau_r < \tau_R) = \left(\frac{r}{x}\right)^{\delta-2} < 1$$

$$\underbrace{\lim_{r\to 0} \lim_{R\to\infty}}_{\text{in this order!!!}} P(\tau_r < \tau_R) = 0$$

(d) $\boxed{\delta = 2}$

$$\lim_{R\to\infty} P_x(\tau_r < \tau_R) = 1$$

$$\lim_{r\to 0} P_x(\tau_r < \tau_R) = 0$$

(e) $\boxed{\delta < 2}$

$$\lim_{r\to 0} P_x(\tau_r < \tau_R) = 1 - \left(\frac{x}{R}\right)^{2-\delta}$$

$$\lim_{R\to\infty} \lim_{r\to 0} P_x(\tau_r < \tau_R) = 1.$$