Balint Toth:

Brownian Motion / 2

Construction of Brownian Motion:

Nevertheless, in a miraculous way, BM exists, as decent mathematical object!!!

Theorem

There exists a (unique) stochastic process

$$t \longmapsto B(t) \qquad t \in [0, \infty)$$

with the properties 1 & 2*/2** & 3.

History: first proof: Norbert Wiener 1923

alternative: Paul Lévy 1948

"invariance principle": P. Erdős & M. Kac 1946

M. Donsker 1952

Soft remarks:

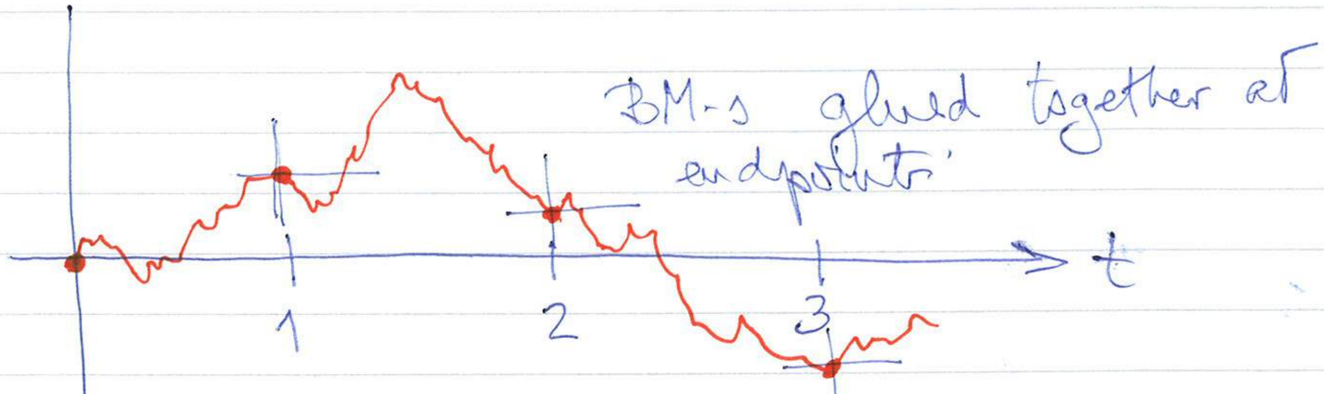
1. sufficient to prove for $t \in [0,1]$:

let $B^{(k)}(\cdot)$, $k = 1, 2, \ldots$ be independent BM-s on $t \in [0,1]$ and

$$t \mapsto \mathcal{B}(t) \qquad t \in [0, \infty)$$

defined as:

$$\mathcal{B}(t) = \sum_{k=1}^{\lfloor t \rfloor} B^{(k)}(1) + B^{(\lfloor t \rfloor + 1)}(t - \lfloor t \rfloor)$$



2. sufficient to consider the $\sigma = 1$ case

$$\underbrace{\tilde{B}(\cdot)}_{\text{variance } \sigma^2} = \sigma \underbrace{B(\cdot)}_{\text{variance } 1}$$

Sketch of N. Wiener's proof: $t \in [0,1]$

<u>Idea</u>: Try expansion with respect to an orthonormal basis in $L^2([0,1])$, with independent Gaussian coefficients.

Let $\{\psi_n(t)\}_{n=1}^{\infty}$ be ortho-normal basis in $L^2([0,1])$

$$\int_0^1 \psi_n(t)\psi_m(t)\,dt = \delta_{n,m}$$

(to be specified later)

Let $(\xi_n)_{n=1}^{\infty}$ be i.i.d. $\mathcal{N}(0,1)$ random variables.

Let $(c_n)_{n=1}^{\infty}$ be real constants (to be specified later)

And write (formally)

$$B(t) = \sum_{n=1}^{\infty} c_n \xi_n \psi_n(t)$$

Hope that $(\psi_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ can be chosen so that

(b) the r.h.s. converges in suff. strong sense

(a) the result has the desired distribution (Gaussian with cov. $\min(t,s)$)

(a)

$$\mathbb{E}(B(t)B(s)) = \mathbb{E}\left(\left(\sum_{n=1}^{\infty} c_n \xi_n \psi_n(t)\right)\left(\sum_{m=1}^{\infty} c_m \xi_m \psi_m(s)\right)\right)$$

$$= \sum_{n,m=1}^{\infty} c_n c_m \psi_n(t)\psi_m(s) \underbrace{\mathbb{E}(\xi_n, \xi_m)}_{= \delta_{n,m}}$$

$$= \sum_{n=1}^{\infty} c_n^2 \psi_n(t)\psi_n(s)$$

$$= \min(t,s)$$

the two must be equal

Define on $L^2([0,1])$ the bounded operator $K \in \mathcal{B}(L^2([0,1]))$

$$Kf(t) := \int_0^1 \mathcal{K}(t,s) f(s)\, ds$$

with kernel $\mathcal{K}(t,s) = \min(t,s)$

$K = K^*$, <u>compact op.</u> (actually: Hilbert-Schmidt)

has same properties as self-adj. matrices

Full set of eigenvectors / eigenvalues:

$$K\psi_n = \lambda_n \psi_n, \quad n = 1, 2, \ldots$$

and $\mathcal{K}(t,s) = \sum_{n=1}^{\infty} \lambda_n \psi_n(t) \psi_n(s)$ see: linear algebra

For this particular kernel

$\lambda_n = \frac{1}{\pi^2}\left(n - \frac{1}{2}\right)^{-2}$, $\psi_n(t) = \sqrt{2} \sin\left(n - \frac{1}{2}\right) t\pi$; $n = 1, 2, \ldots$

HW: check these

good candidate:

$$\text{"}\; B(t) = \sum_{n=1}^{\infty} \frac{1}{\pi}\left(n-\frac{1}{2}\right)^{-1} \xi_n \sqrt{2}\sin\left(n-\frac{1}{2}\right)t\pi \;\text{"}$$

ξ_n i.i.d. $\mathcal{N}(0,1)$

ⓑ check sufficiently strong uniform in t convergence

= difficult =

Remark: Convergence in $L^2([0,1])$ is easy:

$$\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)^{-2} |\xi_n|^2 < \infty \quad \text{almost surely!}$$

also easy: a.s. cvg for fixed t

$$B^N(t) := \sum_{n=1}^{N} c_n \xi_n \psi_n(t) \quad \underline{\text{continuous}}$$

Need: (almost sure) uniform in $t\in[0,1]$ convergence

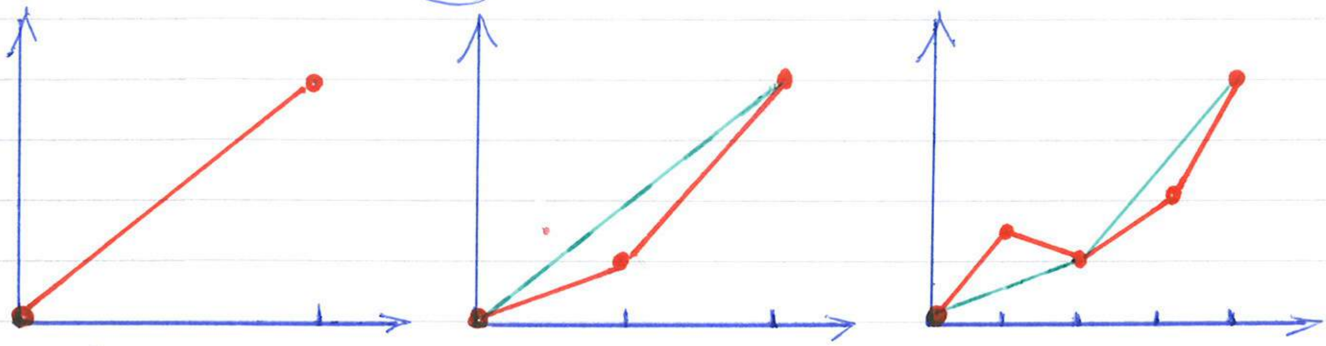
Wiener's proof: uniform cvg. along a well chosen subsequence of N,

hard

$\square$

Paul Lévy's proof (in full detail)

Idea: Sample at diadic rationals $(k 2^{-n})_{k=0}^{2^n}$, and interpolate linearly.



first second third ... approx.

Ingredients

$\xi_1, \xi_{\frac{2k+1}{2^n}}$ $\quad n = 1, 2, \dots$
$\quad k = 0, 1, \dots, 2^{n-1} - 1$

i.i.d. $\mathcal{N}(0,1)$ random variables indexed by diadic rationals in $[0,1]$.

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$$c_n = 2^{-\frac{n+1}{2}} \qquad n = 1, 2, \ldots$$

chosen so that the covariances match

Successive approximation:

$B(0) = 0 \qquad B(1) = \xi_1$

$$B\left(\tfrac{1}{2}\right) = \frac{1}{2}\left(B(0) + B(1)\right) + c_1 \xi_{\frac{1}{2}}$$

$$B\left(\tfrac{1}{4}\right) = \frac{1}{2}\left(B(0) + B\left(\tfrac{1}{2}\right)\right) + c_2 \xi_{\frac{1}{4}}$$

$$B\left(\tfrac{3}{4}\right) = \frac{1}{2}\left(B\left(\tfrac{1}{2}\right) + B(1)\right) + c_2 \xi_{\frac{3}{4}}$$

...

$$B\left(\frac{2k+1}{2^{n+1}}\right) = \frac{1}{2}\left(B\left(\frac{k}{2^n}\right) + B\left(\frac{k+1}{2^n}\right)\right) + c_{n+1} \xi_{\frac{2k+1}{2^{n+1}}}$$

$$k = 0, \ldots, 2^n - 1$$

$B^{(n)}(t)$ obtained by linear interpolation between $B\left(\frac{k}{2^n}\right) : k = 0, 1, \ldots, 2^n$

Proof of almost sure uniform convergence of the sequence $B^{(n)}(t)$: in $t \in [0,1]$

Note that

$$\sup_{0 \le t \le 1} \left| B^{(n+1)}(t) - B^{(n)}(t) \right| = c_{n+1} \max_{0 \le k \le 2^n - 1} \left| \xi_{\frac{2k+1}{2^{n+1}}} \right|$$

$$\mathbb{P}\left(\sup_{0 \le t \le 1} \left| B^{(n+1)}(t) - B^{(n)}(t) \right| > 2^{-\frac{n}{4}} \right) =$$

$$\mathbb{P}\left(\max_{0 \le k \le 2^{n-1}} \left| \xi_{\frac{2k+1}{2^{n+1}}} \right| > 2^{\frac{n+2}{4}} \right) \le$$

$$2^n \, \mathbb{P}\left(|\xi| > 2^{\frac{n+2}{4}} \right) \le$$

where $\xi \sim \mathcal{N}(0,1)$

$$\sqrt{\frac{2}{\pi}} \cdot 2^n \exp\left(-2^{\frac{n}{2}}\right) \leftarrow$$

this is summable

By Borel-Cantelli: almost surely $\exists\, N = N(\omega)$ (random) such that

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for $n \geq N$

$$\sup_{0 \le t \le 1} \left| B^{(n+1)}(t) - B^{(n)}(t) \right| \le 2^{-\frac{n}{4}}$$

Hence: the sequence of functions

$$t \longmapsto B^{(n)}(t)$$

is uniformly (in $t \in [0,1]$) convergent $\square$

<u>Standard BM in $\mathbb{R}^d$:</u>

$t \longmapsto B(t) \in \mathbb{R}^d$

$B(t) = \left(B_1(t), B_2(t), \dots, B_d(t) \right)$

where $\left(B_j(t) \right)_{j=1}^d$ are independent 1d Brownian Motions

The distribution of $t \mapsto B(t)$ is invariant under orthogonal transformations (rotations) of $\mathbb{R}^d$