Ito calculus / 1

Baliné Tibor: Stochastic integration, the Itô integral

Motivation:

A typical SDE:

$$" \dot{X}(t) = b(t) + \sigma(t)\,\dot{w}(t) "$$

where:

- $b(t) = b(t,\omega)$, $\sigma(t) = \sigma(t,\omega)$
  $w(t) = w(t,\omega)$, $X(t) = X(t,\omega)$
  are random / stochastic processes defined on the same filtered probab. space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ and are adapted to the filtration
- $t \mapsto b(t)$, $t \mapsto \sigma(t)$ are

continuous (slightly less regularity would be sufficient)

- $b(t)$, $\sigma(t)$ may also depend on $(X(s) : 0 \leq s < t)$!

- $\dot{w}(t)$ = random driving
  assumed: stationary + "white"
  "white" = uncorrelated, zero mean

$$E(\dot{w}(t)) = 0, \quad E(\dot{w}(t)\dot{w}(s)) = \delta(t-s)$$

- $b(t)$ = instantaneous drift
  $\sigma(t)$ = instantaneous dispersion

What is the "white noise" $\dot{w}(t)$?

$$\mathbb{E}(\dot{w}(t)) = 0, \quad \mathbb{E}(\dot{w}(s)\dot{w}(t)) = \delta(t-s)$$

+ Gaussian

looks like: $\boxed{\dot{w}(t) = \frac{dB(t)}{dt}}$

doesn't exist as a decent process.

Better rewrite: $\dot{w}(t)\,dt = dB(t)$

The SDE re-written:

$$dX(t) = b(t)\,dt + \sigma(t)\,dB(t)$$

Integrated:

$$X(t) = X(0) + \underbrace{\int_0^t b(s)\,ds}_{\text{Riemann integral}} + \underbrace{\int_0^t \sigma(s)\,dB(s)}_{?}$$

3b

Examples of SDE-s

① Linear $\quad b(t) = r \cdot X(t) \quad \sigma(t) = a \cdot X(t)$

$$dX(t) = \underbrace{\overset{\text{interest rate}}{r}\, X(t)\,dt}_{} + \underbrace{a\,X(t)}_{}\, dB(t)$$

in finance $\quad (X(t) \geq 0)$

sln: "geometric BM"

② Langevin's eq: $\quad b(t) = -\gamma V(t) \quad \sigma(t) = \sigma$

$$\overset{\text{velocity of a particle}}{dV(t)} = \underbrace{-\gamma V(t)\,dt}_{\text{friction / damping}} + \underbrace{\sigma\, dB(t)}_{\text{random forcing}}$$

in physics

sln "Ornstein-Uhlenbeck process

③ $\quad dX(t) = X(t)(1-X(t))\,dB(t); \quad X(t) \in (0,1)$

in population dynamics $\quad \sigma(t) = \sqrt{X(t)(1-X(t))}$

$b(t) = 0$

sln "Fisher-Wright process"

(3c)

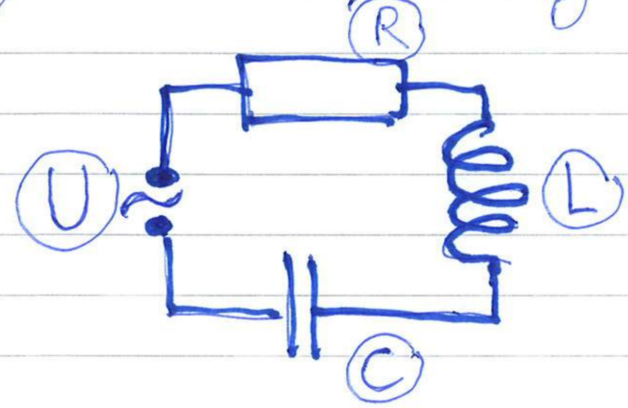
(4) An example from electric circuits

$$L\ddot{Q}(t) + R\dot{Q}(t) + C^{-1}Q(t) = U(t)$$

$Q(t)$ = charge ; $\dot{Q}(t)$ = current ; $\ddot{Q}(t)$ =

L = impedance ; R = resistence ; C = capacity

$U(t)$ = tension / voltage / potential difference



Randomized voltage:

$U(t) = F(t) \quad \dot{W}(t)$

call $X_1 = Q$, $X_2 = \dot{Q}$

$$\dot{X}_1 = X_2 \ ; \ \dot{X}_2 = -\frac{R}{L}X_2 - \frac{1}{CL}X_1 + \frac{1}{L}F + \frac{\alpha}{L}\dot{W}$$

$$dX(t) = -GX(t)dt + H(t)dt + K\,dB(t)$$

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} ; \ H(t) = \begin{pmatrix} 0 \\ L^{-1}F(t) \end{pmatrix} ; \ K = \begin{pmatrix} 0 \\ \alpha L^{-1} \end{pmatrix} ; \ G = \begin{pmatrix} 0 & -1 \\ (CL)^{-1} & RL^{-1} \end{pmatrix}$$

What is the "stochastic integral" $\int_0^t v(s)\, dB(s)$?

Try Riemann sums

$$\int_0^t v(s)\, dB(s) \overset{?}{=} \lim \sum_i v(s_i^*)\big(B(s_{i+1}) - B(s_i)\big)$$

The limit may not exist since $B(\cdot)$ has no bdd. variation.

(However: if $v(\cdot)$ has bdd variation, then partial summation helps. But typically this is not the case!)

An instructive (Counter) example:

$$\int_0^t B(s)\, dB(s) \overset{?}{=} \frac{1}{2} B(t)^2$$

do it in two ways:

1. $$\sum_n^{(1)} := \sum_i B(t_i)\big(B(t_{i+1}) - B(t_i)\big)$$

2. $$\sum_n^{(2)} := \sum_i B(t_{i+1})\big(B(t_{i+1}) - B(t_i)\big)$$

$$\sum_n^{(2)} - \sum_n^{(1)} = \sum_i \big(B(t_{i+1}) - B(t_i)\big)^2 \longrightarrow t$$

see: quadratic variation of BM

The Itô choice: $t_i^* = t_i$

The Stratonovich choice: $t_i^* = (t_i + t_{i+1})/2$

The Itô integral – definition/construction:

Ingredients:

$(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ filtered probab. space.

$t \longmapsto B(t)$ BM on it,

- adapted: $\mathcal{F}_t^B \subseteq \mathcal{F}_t$
- martingale wrt $\mathcal{F}_t$.

Definition: "progressively measurable process"

$y(t)$ stoch. process on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

such that

$(\forall t)$ $(s,\omega) \longmapsto y(s,\omega)$ $0 \leq s \leq t$

is jointly $(\underset{\text{Borel } \sigma\text{-alg on } \mathbb{R}_+}{\mathcal{B}} \times \mathcal{F}_t)$ - measurable

(slightly more than just adapted)

$0 < T < \infty$ fixed.

The class $\mathcal{V}_T$: $\varphi : [0,T] \times \Omega \to \mathbb{R}$

① φ progressively measurable

② $\|\varphi\|^2 := \mathbb{E}\left(\int_0^T |\varphi(t,\omega)|^2 dt \right) < \infty$.

$\mathcal{L}^2(\Omega \times [0,T], \text{Progr meas.}, dP \times ds)$ 7

$(\mathcal{V}_T, \|\cdot\|)$ is a Hilbert space.

$$\mathcal{V}_T^s := \Big\{ \varphi \in \mathcal{V}_T : \varphi(t,\omega) = \sum_{i=1}^{n} \varphi_i(\omega) \chi_{[t_{i-1}, t_i)},$$

$$0 = t_0 < t_1 < \dots < t_n = T, \ \varphi_i \ \mathcal{F}_{t_i}\text{-measurable} \Big\}$$

"Simple" processes, piecewise constant.

The partition is deterministic / doesn't depend on $\omega \in \Omega$!

For $\varphi \in \mathcal{V}_T^s$ define

$$\int_0^T \varphi(s)\, dB(s) := \sum_{i=1}^{n} \varphi_{i-1} \big(B(t_i) - B(t_{i-1}) \big)$$

Let $I : (\mathcal{V}_T^s, \|\cdot\|^2) \longrightarrow L^2(\Omega, \mathcal{F}_T, P)$

$$I(\varphi) := \int_0^T \varphi(s)\, dB(s)$$

Lemma 1: I is an isometry:

$$\mathbb{E}\left(\left(\int_0^T \varphi(s)\, dB(s)\right)^2\right) = \mathbb{E}\left(\int_0^T \varphi(s)^2\, ds\right) = \|\varphi\|^2$$

Proof:

$$\mathbb{E}\left(\left(\int_0^T \varphi(s)\, dB(s)\right)^2\right) =$$

$$\mathbb{E}\left(\sum_i \varphi_{i-1}\left(B(t_i) - B(t_{i-1})\right)\right)^2 =$$

$$\sum_i \mathbb{E}\left(\varphi_{i-1}^2 \left(B(t_{i-1}) - B(t_i)\right)^2\right) +$$

$$2 \sum_{i<j} \underbrace{\mathbb{E}\left(\varphi_{i-1}\varphi_{j-1}\left(B(t_i) - B(t_{i-1})\right)\left(B(t_j) - B(t_{j-1})\right)\right)}_{= 0 \text{ (indep. increments of } B)}$$

$$= \dots = \sum_i (t_i - t_{i-1})\, \mathbb{E}(\varphi_{i-1}^2) = \|\varphi\|^2.$$

$\square$

Lemma 2 $\mathcal{V}_T^S$ is dense in $(\mathcal{V}_T, \|\cdot\|)$

Proof: standard (but tedious...)

Theorem

$\int_0^T \varphi(t)\, dB(s)$ extends continuously from $(\mathcal{V}_T^S, \|\cdot\|)$ to $(\mathcal{V}_T, \|\cdot\|)$.

That is: for $\varphi \in \mathcal{V}_T$

$$\int_0^T \varphi(s)\, dB(s) := \lim_{n\to\infty} \int_0^T \varphi_n(s)\, dB(s)$$

where $\varphi_n \in \mathcal{V}_T^S$, $\|\varphi_n - \varphi\| \longrightarrow 0$.

The limit (in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$) doesn't depend on the approximating sequence φ_n.

$$E\left(\left(\int_0^T \varphi(s)\, dB(s)\right)^2\right) = E\left(\int_0^T \varphi(s)^2\, ds\right)$$

Some basic properties of Itô integral:

$0 \leq S \leq T \leq U < \infty$:

① $\int_S^T \varphi(s)\, dB(s)$ is $\mathcal{F}_T$ measurable

② $\int_S^U \varphi(s)\, dB(s) = \int_S^T \varphi(s)\, dB(s) + \int_T^U \varphi(s)\, dB(s)$

③ a, b deterministic numbers

$$\int_S^T (a\varphi(s) + b\psi(s))\, dB(s) = a\int_S^T \varphi(s)\, dB(s) + b\int_S^T \psi(s)\, dB(s)$$

④ $$E\left(\int_S^T \varphi(s)\, dB(s)\right) = 0$$

$$E\left(\left(\int_S^T \varphi(s)\, dB(s)\right)^2\right) = E\left(\int_S^T \varphi(s)^2\, ds\right).$$

Examples: 1 $\int_0^T B(s)\,dB(s) = \frac{1}{2}\left(B(T)^2 - T\right)$

HW: Hint: $\varphi(s) = B(s)$

$t_i = i\frac{T}{n} \qquad 0 \leq i \leq n$

$$\varphi_n(s) = \sum_{i=1}^{n} B(t_{i-1})\,\chi_{[t_{i-1},t_i)}(t)$$

Step 1 $\quad \|\varphi - \varphi_n\|^2 \longrightarrow 0$

Step 2 $\quad$ compute $L^2\text{-}\lim_{n\to\infty} \int_0^T \varphi_n(s)\,dB(s)$.

2 $\quad \int_0^T B(s)^2\,dB(s) = \frac{1}{3}B(t)^3 - \int_0^T B(s)\,ds$

HW similarly

Question:

does a (sufficiently regular)

$$t \longmapsto \int_0^t \varphi(s)\, dB(s)$$

$\boxed{\text{process}}$ exist on the probability sp.

$\left(\Omega, (\mathcal{F}_t)_t, \mathbb{P}\right)$?

Mind $\int_0^t \varphi(s)\, dB(s)$ was defined as L^2 limit, for t fixed. It's a.s. defined for t fixed or for countably many $(t_j)_{j\geq 0}$. They are a priori not a.s. simultaneously defined for all $t \in [0,T]$.

<u>Theorem</u> : Let $\varphi \in \mathcal{V}_T$

There exists an adapted process

$t \longmapsto \mathcal{J}_\varphi(t)$ with a.s. continuous

sample paths such that

$$\forall t \in [0,T] : \mathbb{P}\left(\int_0^t \varphi(s)\, dB(s) = \mathcal{J}(t) \right) = 1 .$$

The process

$t \longmapsto \mathcal{J}_\varphi(t)$ is actually a

martingale and $\forall \lambda > 0$:

$$\mathbb{P}\left(\sup_{0 \le t \le T} \left| \mathcal{J}_\varphi(t) \right| \geq \lambda \right) \leq \lambda^{-2}\, \mathbb{E}\left(\int_0^T \varphi(s)^2\, ds \right)$$

Proof: Choose $\varphi_n \in V_T^S$ so that

$$\|\varphi - \varphi_n\|^2 = \mathbb{E}\left(\int_0^T (\varphi(s) - \varphi_n(s))^2 ds\right) < 2^{-3n}$$

$$\mathcal{I}_n(t) := \int_0^t \varphi_n(s)\, dB(s) \quad \text{continuous martingale}$$

(HW)

By Doob's inequality + Schwarz's inequality

$$\mathbb{P}\left(\sup_{0\le t\le T} |\mathcal{I}_{n+1}(t) - \mathcal{I}_n(t)| > 2^{-n}\right) \le$$

$$2^{2n}\, \mathbb{E}\left(\int_0^T (\varphi_{n+1}(s) - \varphi_n(s))^2 ds\right) \le C \cdot 2^{-n}$$

By Borel-Cantelli a.s. $\exists\, n_0 = n_0(\omega)$ s.t. for $n \ge n_0$:

$$\sup_{0\le t\le T} |\mathcal{I}_n(t) - \mathcal{I}_{n+1}(t)| \le 2^{-n}$$

$$\Rightarrow \text{a.s.} \quad \mathcal{I}_n(t) \xrightarrow[t\in[0,T]]{\text{uniformly}} \mathcal{I}(t)$$

$\mathcal{I}(t)$ continuous martingale

$$(\forall t) \quad \mathcal{I}(t) = \int_0^t \varphi(s)\, dB(s)$$

$\square$

The multidimensional Itô-integral

$(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

$B(t) = (B_1(t), \dots, B_k(t)) \in \mathbb{R}^k$

k-dimensional BM, adapted to $(\mathcal{F}_t)$

$\varphi = (\varphi_1, \dots, \varphi_k)$; $\varphi_j \in \mathcal{V}_T$

$$\int_0^t \varphi(s) \cdot dB(s) := \sum_{j=1}^{k} \int_0^t \varphi_j(s)\, dB_j(s).$$