Stochastic Processes
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Homework sheet 1

1.1 Let i and j be two states in the state space of a discrete time, discrete state space, time homogeneous Markov chain. Show that if i and j communicate (that is: it's possible to get from i to j and also from j to i), then their periods are the same.

1.2 Let X_n be a discrete time, discrete state space, time homogeneous Markov chain. A state i is said to be *recurrent* if $\mathbb{P}(\exists n > 0 : X_n = i \mid X_0 = i) = 1$. (Otherwise it is called transient.) Show that if i and j communicate and i is recurrent, then j is also recurrent.

1.3 Let X_n be a discrete time, discrete state space, time homogeneous Markov chain. A recurrent state i is said to be *positive recurrent* if there exists a stationary distribution π for which $\pi_i > 0$. (Otherwise it is called null-recurrent.) Show that if i and j communicate and i is positive recurrent, then j is also positive recurrent.

1.4 Let P be the transition matrix of a discrete time, finite state space, time homogeneous Markov chain. Show that if the Markov chain is irreducible and aperiodic, then there is an n for which all elements of P^n are positive.

1.5 Let P be the transition matrix of a discrete time, finite state space, time homogeneous Markov chain. We have seen (or will see) that if all elements of P are positive, then the Markov chain has a unique invariant distribution. Show (as a consequence) that the same is true if the Markov chain is irreducible and aperiodic.

1.6 Let X_n be discrete time, time homogeneous Markov chain on the discrete state space S. Let P be its transition matrix, so $P_{i,j} = \mathbb{P}(X_{n+1} = j | X_n = i)$. Let $\tau \in \mathbb{N}$ be a random time.

a.) Show that if τ is a stopping time (for the natural filtration of X_n), then

$$\mathbb{P}(X_{\tau+1} = j | X_\tau = i) = P_{i,j}. \tag{1}$$

(This is the simplest form of the *strong Markov property*).

b.) Give a specific example of a Markov chain X_n and a random time τ, which is not a stopping time, such that (1) does not hold.

1.7 *(ON-OFF process)* Consider the discrete time, time homogeneous Markov chain on the state space $S = \{0, 1\}$, with the transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

(This models a system which can either be ON or OFF at each time moment. It is turned ON with probability α whenever it is OFF, and it is turned OFF with probability β whenever it is ON.)

Calculate $\mathbb{P}(X_n = j | X_0 = i)$ explicitly, for every i, j and n.

1.8 Let X_n and Y_n be independent time homogeneous Markov chains on the state space $\{1, 2, 3\}$, with the same transition matrix

$$P = \begin{pmatrix} 0.1 & 0.4 & 0.5 \\ 0.2 & 0.6 & 0,2 \\ 0,3 & 0.1 & 0.6 \end{pmatrix}$$

(but possibly different initial distributions). Let $T = \inf\{n \in \mathbb{N} | X_n = Y_n\}$ be the first time when the two systems meet. Show that $\mathbb{E}T < \infty$ and give an easy (finite) upper estimate.

1.9 Let X_n be a time homogeneous Markov chain on the state space $S = \mathbb{N}$, with transition matrix P such that $P_{i,j} = 0$ whenever $|j - i| > 1$. (This is called a birth and death process.) Find the invariant measures in the cases below. (The invariant measures may or may not be finite. If they are finite, normalize them to get an invariant distribution.)

a.) $P_{i,i+1} = \frac{1}{3}$ for all i and $P_{i,i-1} = \frac{2}{3}$ for all $i > 0$ (so $P_{0,0} = \frac{2}{3}$ and $P_{i,i} = 0$ for $i < 0$)

b.) $P_{i,i+1} = \frac{2}{3}$ for all i and $P_{i,i-1} = \frac{1}{3}$ for all $i > 0$ (so $P_{0,0} = \frac{1}{3}$ and $P_{i,i} = 0$ for $i < 0$)

1.10 (*coupon collector*) We keep drawing cards one by one from a stack on N cards at random, *with replacement*, meaning that we always put the drawn card back to the stack and shuffle it.

a.) Let X_n be the number of cards seen up to (and including) the nth draw. Show that X_n is a Markov chain and give the transition probabilities.

b.) For $k = 0, 1, \ldots, N$ let T_k be the number of draws needed until k different cards are seen. Show that T_k is a Markov chain and give the transition probabilities.