Stochastic Processes
CEU Budapest, winter semester 2018/19
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Homework sheet 2

2.1 Consider the Markov chain in Exercise 1.9. For every $i \in \mathbb{N}$, calculate the expected first hitting time of 0 starting from i.

2.2 A monkey is sitting at a typewriter with keys only for the 26 letters of the English alphabet, pressing keys completely at random (one at at time). You are watching to see when the character sequence ABRACADABRA shows up. For $n \in \mathbb{N}$ let $X_n \in S := \{0, 1, \dots, 11\}$ be the length of the longest *initial* subsequence of ABRACADABRA that matches the last keys pressed by the monkey. For example, if she presses FJOIRFCABRACACABF..., then $X_0 = 0$, $X_{11} = 4$, $X_{15} = 1$. Let $\tau = \inf\{n \in \mathbb{N} \,|\, X_n = 11\}$ (the first hitting time of the state 11).

a.) Give the transition matrix of the Markov chain X_n. (This is a 12 by 12 matrix, so it's a good idea to list the nonzero elements only.)

b.) Show that τ is almost surely finite.

c.) For $i \in S$ let $e_i = \mathbb{E}(\tau \,|\, X_0 = i)$. Write a system of linear equations for the a_i and find e_0 using it.

d.) For $i \in S$ and $z \in [0, 1]$ let $g_i(z) = \mathbb{E}\left(z^\tau \,|\, X_0 = i\right)$. Write a system of linear equations for the $g_i(z)$ and find $g_0(z)$ using it.

2.3 The Euler gamma function is defined for $s > 0$ by $\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} \, \mathrm{d}x$. The beta function is defined for $s, t > 0$ by $B(s, t) := \int_0^1 x^{s-1}(1-x)^{t-1}$. (The symbol is meant to be a capital greek beta.)

A random variable has Γ distribution with shape parameter $s > 0$ and rate $\lambda > 0$ (denoted as $\Gamma(s, \lambda)$) if it has density

$$f_{s,\lambda}(x) = \begin{cases} \frac{\lambda^s}{\Gamma(s)} x^{s-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if not} \end{cases}.$$

Let $s, t, \lambda > 0$. Let $X \sim \Gamma(s, \lambda)$ and $Y \sim \Gamma(t, \lambda)$ be independent and let $Z = X + Y$. Calculate the (densitiy of) the distribution of Z. As a side result, show that $B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$.

2.4 Let $\xi_1, \xi_2, \dots$ be independent random variables having the same exponential distribution with parameter λ (meaning expectation $1/\lambda$) and let $\tau_n = \xi_1 + \dots + \xi_n$.

a.) Calculate the distribution of τ_n. (Calculate the density or the distribution function, as you like.)

b.) For every $t \in \mathbb{R}^+$, define $X_t = \sup n : \tau_n \leq t$. For each n and t, calculate the probability $\mathbb{P}(X_t \geq n)$.

c.) What is the distribution of X_t?

d.) For $a \leq b \in \mathbb{R}^+$ let $X_{[a,b]} = \#\{n \,|\, \tau_n \in [a, b]\}$. Calculate the distribution of $X_{[a,b]}$.

e.) Let $[a_1, b_1], [a_2, b_2], \dots, [a_k, b_k]$ be disjoint intervals in $\mathbb{R}^+$. Describe the joint distribution of $(X_{[a_1,b_1]}, X_{[a_2,b_2]}, \dots, X_{[a_k,b_k]})$.

2.5 Let X and Y be independent random variables, $X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$. What is the distribution of $Z := X + Y$?

2.6 Joe generates a random variable N with $Poi(\lambda)$ distribution. Then he rolls a die N times. (The die is biased, so the ith face turning up has probability p_i in each roll.) Let X_i be the number of rolls when the ith face turns up.

a.) Calculate the distribution of X_1.

b.) Describe the joint distribution of $(X_1, \dots, X_k)$.

2.7 For every $k \in \mathbb{N}$ let $N_k \sim Poi(\lambda)$ and let the N_k be independent. On each unit interval $[k, k+1)$ we place N_k points, uniformly, independently of each other and the values $\{N_i\}$. (More precisely: conditionally independently, given N_k.)

a.) Let τ_1 be the place of the leftmost point. Calculate the distribution of τ_1.

b.) Let τ_2 be the place of the second leftmost point. Calculate the distribution of $\xi_2 = \tau_2 - \tau_1$.

c.) Describe the joint disjoint of (τ_1, ξ_2).

d.) Let $X_{[a,b]}$ be the number of points in the interval $[a, b]$. What is the distribution of $X_[a, b]$?

e.) What is the joint distribution of $(X_{[a_1,b_1]}, X_{[a_2,b_2]}, \dots, X_{[a_k,b_k]})$ if $[a_1, b_1], [a_2, b_2], \dots, [a_k, b_k]$ are disjoint intervals in $\mathbb{R}^+$?

2.8 At an ice cream stand customers arrive one by one at random times given by a Poisson process of rate $\lambda > 0$ and join the queue. They are served one by one, the serive times are independent and have $Exp(\mu)$ distribution with $\mu > 0$. Let $X(t) \in S = \mathbb{N}$ be the length of the queue at time t.

a.) Givee the infinitesimal generator of the Markov chain $X(t)$.

b.) Give the transition matrix of the built-in discrete time Markov chain.

c.) Find the stationary distributions of $X(t)$, and also of the built-in discrete time Markov chain. Compare!

d.) Assume that at $X(t) = 1$. What is the expected number of costumers served until the queue first becomes empty (so that the ice cream man can take a rest)?

e.) Assume that at $X(t) = 1$. What is the expected time elapsed until the queue first becomes empty (so that the ice cream man can take a rest)?

2.9 In a corridor with no windows, there are 3 light bulbs working all the time (unless they are burnt out). Each bulb, independently of the others, burns out after an exponentially distributed random time with expectation $\frac{1}{2}$ year. The janiter comes occasionally, once a year on average, according to a Poisson process. When he comes, he replaces every burnt bulb. Let $X(t)$ be the number of bulbs that work at time t.

a.) Give the infinitesimal generator of $X(t)$. Be careful!

b.) Give the transition matrix of the built-in discrete time Markov chain.

c.) On the long run, in what percentage of the time is the corridor totally dark?

2.10 Bob has 3 light bulbs in his room. Each bulb, independently of the others, burns out after an exponentially distributed random time with expectation $\frac{1}{2}$ year. Bob doesn't care, and does not replace them until the all burn out, but then he replaces them all immediately. Let $X(t)$ be the number of bulbs that work at time t.

a.) Give the infinitesimal generator of $X(t)$. Be careful!

b.) Give the transition matrix of the built-in discrete time Markov chain.

c.) On the long run, in what percentage of the time is there exactly 1 bulb that works?

2.11 In a radiactive sample consisting of many particles, a certain kind of nucleus (called "SP particle") is produced according to a Poisson process, at a rate of 10 particles per second. Each particle, independently of everything else, lives for an exponentially distributed random time with expectation 5 seconds, after which it decays, emitting an electron that we can observe.

This process has already been going on for a long time.

a.) Pick a fixed time interval of length 1 second. What is the probability that in this time interval no decay of an SP particle can be observed?

b.) Pick a fixed time moment. What is the expected number of SP particles that exist at this given time moment?

c.) Pick a fixed time moment. What is the expected number of SP particles that exist at this given time moment, and are more than 50 seconds old?

d.) Pick a fixed time moment. What is the expected number of SP particles that exist at this given time moment?

e.) Let $X(t)$ be the number of SP particles that exist at time t. Describe X as a continuous time Markov chain. Give the transition rates and calculate the stationary distribution.