Stochastic Processes
CEU Budapest, winter semester 2013/14
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Homework sheet 1

1. The random vector $Y = (Y_1, Y_2, \dots, Y_n)^T$ (think of it as a column vector) is said to have multivariate normal distribution if there are independent standard normal random variables $X = (X_1, X_2, \dots, X_k)^T$ (think of this as a column vector as well), a vector $b \in \mathbb{R}^n$ and an $n \times k$ real matrix A such that $Y = b + AX$.

   a.) Show that if $Y = (Y_1, Y_2, \dots, Y_n)^T$ is multivariate normal, K is an $m \times n$ real matrix and $c \in \mathbb{R}^m$, then $Z := c + KY$ is also multivariate normal.

   b.) Calculate the expectation vector m and the covariance matrix C of the random vector Y. (Meaning $m_i = \mathbb{E}Y_i$ and $C_{ij} = Cov(X_i, X_j)$.)

   c.) Suppose that $n = k$ and A is invertible. Calculate the (joint) density of Y (w.r.t. Lebesgue measure on $\mathbb{R}^n$). Conclude that a multivariate normal distribution is characterized entirely by its expectation vector and covariance matrix. (This is true in general, but this calculation only shows it in the "nondegenerate" case.)

2. Consider the family of measures

$$H = \{\mu_{t_1,t_2,\dots,t_n} \,|\, n \in \mathbb{N},\ 0 \le t_k \le 1 \text{ for every } k = 1, 2, \dots, n\},$$

   where for every $n \in \mathbb{N}$ and every $t_1, \dots, t_n \in [0, 1]$, $\mu_{t_1,t_2,\dots,t_n}$ is the n-dimensional normal distribution on $\mathbb{R}^n$ with mean zero and covariance matrix elements $C_{ij} = \min t_i, t_j - t_i t_j$.

   Show that there exists a (real-valued) stochastic process on the timne interval $[0, 1]$ with this family of measures H as its finite dimensional marginals.

3. Let $I = [0, 1]$ and consider the (uncountably infinite) product set $\Omega := \mathbb{R}^I$, which is nothing else than the set of all real valued functions on $[0, 1]$, that is, $\Omega = \{f : I \to \mathbb{R}\}$. Let $\mathcal{F}$ be the product sigma-algebra on Ω, where each factor $\mathbb{R}$ was equipped with the Borel sigma-algebra.

   Show that the set of continuous functions is not measurable:

$$\{f : I \to \mathbb{R} \,|\ f \text{ is continuous}\} \notin \mathcal{F}.$$

4. Let $\xi_1, \xi_2, \dots$ be independent random variables having the same exponential distribution with parameter λ (meaning expectation $1/\lambda$) and let $\tau_n = \xi_1 + \dots + \xi_n$.

   a.) Calculate the distribution of τ_n. (Calculate the density or the distribution function, as you like.)

   b.) For every $t \in \mathbb{R}^+$, define $X_t = \sup n : \tau_n \le t$. For each n and t, calculate the probability $\mathbb{P}(X_t \ge n)$.

   c.) What is the distribution of X_t?

   d.) For $a \le b \in \mathbb{R}^+$ let $X_{[a,b]} = \#\{n \,|\, \tau_n \in [a, b]\}$. Calculate the distribution of $X_{[a,b]}$.

   e.) Let $[a_1, b_1], [a_2, b_2], \dots, [a_k, b_k]$ be disjoint intervals in $\mathbb{R}^+$. Describe the joint distribution of $(X_{[a_1,b_1]}, X_{[a_2,b_2]}, \dots, X_{[a_k,b_k]})$.

5. Let X and Y be independent random variables, $X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$. What is the distribution of $Z := X + Y$?

6. Joe generates a random variable N with $Poi(\lambda)$ distribution. Then he rolls a die N times. (The die is biased, so the ith face turning up has probability p_i in each roll.) Let X_i be the number of rolls when the ith face turns up.

   a.) Calculate the distribution of X_1.

   b.) Describe the joint distribution of $(X_1, \dots, X_k)$.

7. For every $k \in \mathbb{N}$ let $N_k \sim Poi(\lambda)$ and let the N_k be independent. On each unit interval $[k, k+1)$ we place N_k points, uniformly, independently of each other and the values $\{N_i\}$. (More precisely: conditionally independently, given N_k.)

   a.) Let τ_1 be the place of the leftmost point. Calculate the distribution of τ_1.

   b.) Let τ_2 be the place of the second leftmost point. Calculate the distribution of $\xi_2 = \tau_2 - \tau_1$.

   c.) Describe the joint disjoint of (τ_1, ξ_2).

   d.) Let $X_{[a,b]}$ be the number of points in the interval $[a, b]$. What is the distribution of $X_[a, b]$?

   e.) What is the joint distribution of $(X_{[a_1,b_1]}, X_{[a_2,b_2]}, \dots, X_{[a_k,b_k]})$ if $[a_1, b_1], [a_2, b_2], \dots, [a_k, b_k]$ are disjoint intervals in $\mathbb{R}^+$?

8. Let i and j be two states in the state space of a discrete time, discrete state space, time homogeneous Markov chain. Show that if i and j communicate (that is: it's possible to get from i to j and also from j to i), then their periods are the same.

9. Let X_n be a discrete time, discrete state space, time homogeneous Markov chain. A state i is said to be *recurrent* if $\mathbb{P}(\exists n > 0 : X_n = i \,|\, X_0 = i) = 1$. (Otherwise it is called transient.) Show that if i and j communicate and i is recurrent, then j is also recurrent.

10. Let X_n be a discrete time, discrete state space, time homogeneous Markov chain. A recurrent state i is said to be *positive recurrent* if there exists a stationary distribution π for which $\pi_i > 0$. (Otherwise it is called null-recurrent.) Show that if i and j communicate and i is positive recurrent, then j is also positive recurrent.

11. Let P be the transition matrix of a discrete time, finite state space, time homogeneous Markov chain. Show that if the Markov chain is irreducible and aperiodic, then there is an n for which all elements of P^n are positive.

12. Let P be the transition matrix of a discrete time, finite state space, time homogeneous Markov chain. We have seen that if all elements of P are positive, then the Markov chain has a unique invariant distribution. Show (as a consequence) that the same is true if the Markov chain is irreducible and aperiodic.