Stochastic Processes
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Homework sheet 2

1. Let $t \mapsto X(t)$ be a stochastic process in a complete separable metric space S. Prove that the following two formulations of the Markov property are actually equivalent. (Note that formulation (b) is a priori stronger than (a).)

   (a) For any $0 \le t$, $0 \le u$ and $F : S \to \mathbb{R}$ bounded and measurable
   $$\mathbb{E}(F(X(t+u)) \,|\, \mathcal{F}_t^X) = \mathbb{E}(F(X(t+u)) \,|\, \sigma(X(t))).$$

   (b) For any $0 \le t$, $n \in N$, $0 \le u_1 \le u_2 \le \dots \le u_n$ and $F : S^n \to \mathbb{R}$ bounded and measurable
   $$\mathbb{E}(F(X(t+u_1), X(t+u_2), \dots, X(t+u_n)) \,|\, \mathcal{F}_t^X) =$$
   $$= \mathbb{E}(F(X(t+u_1), X(t+u_2), \dots, X(t+u_n)) \,|\, \mathcal{F}(X(t))).$$

2. Let W_t be a Wiener process. Prove that $t \mapsto W(t)$ is a martingale and $t \mapsto W(t)^2$ is a submartingale (with respect to the filtration $(\mathcal{F}_t^W)_{t \le 0}$).

3. Let $t \mapsto M(t)$ be a martingale (w.r.t. a filtration $(\mathcal{F}_t)_{t \ge 0}$) and $\psi : \mathbb{R} \to \mathbb{R}$ a convex function. Let $Y(t) := \psi(M(t))$. Assuming that $\mathbb{E}(|\psi(M(t))|) < \infty$ for all $t \ge 0$, prove that $t \mapsto Y(t)$ is a submartingale.

4. Show that if W is a Wiener process, then the processes $t \mapsto W(t)$, $t \mapsto W(t)^2 - t$ and $t \mapsto W(t)^3 - 3tW(t)$ are martingales adapted to the natural filtration of W.

5. Let W and W_1 be two independent Wiener processes. Check whether the following processes are martingales with respect to the natural filtration:

   (a) $X(t) = W(t) + 4t$,
   (b) $X(t) = W(t)^2$,
   (c) $X(t) = t^2 W(t) - 2\int_0^t sW(s)\,\mathrm{d}s$,
   (d) $X(t) = W(t)W_1(t)$.

6. Let $s \mapsto v(s)$ be a smooth (say, twice continuously differentiable) deterministic function with $\sup_{0 \le s \le T} |v'(s)| \le C$. Prove directly from the definition of the Itô integral that (for $t \in [0, T]$)
   $$\int_0^t v(s)\,\mathrm{d}W(s) = v(t)W(t) - \int_0^t v'(s)W(s)\,\mathrm{d}s.$$
   Hint: Write
   $$v(s_{i+1})W(s_{i+1}) - v(s_i)W(s_i) = v(s_i)(W(s_{i+1}) - W(s_i)) + W(s_{i+1})(v(s_{i+1})) - v(s_i)).$$

7. Prove directly form the definition of the Itô integral that
   $$\int_0^t W(s)\,\mathrm{d}W(s) = \frac{1}{2}W(t)^2 - \frac{t}{2}$$
   and
   $$\int W(s)^2\,\mathrm{d}W(s) = \frac{1}{3}W(t)^3 - \int^t W(s)\,\mathrm{d}s.$$

8. Let $v, w : [0, T] \to \mathbb{R}$ be progressively measurable and square integrable w.r.t. the product measure. Let $C, D \in \mathbb{R}$. Suppose that

$$\int_0^T v(s)\,\mathrm{d}W(s) + C = \int_0^T w(s)\,\mathrm{d}W(s) + D.$$

Show that $C = D$ and $v = w$ (s, ω)-almost everywhere.