Stochastic Processes
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Homework sheet 3

1. Let W_t be a Wiener process and let $f : [0,T] \to \mathbb{R}$ be a deterministic square integrable function (w.r.t. Lebesgue measure, i.e. $\int_0^T f(t)\,\mathrm{d}t < \infty$). Show that the random variable $X := \int_0^T f(t)\,\mathrm{d}W_t$ is Gaussian. What are its expectation and variance? *(Hint: use the definition of the integral and approximate f with simple functions. If you like, you can (but don't need to) assume that f is continuous.)*

2. Let W_t be a Wiener process and let $f : [0,T] \to \mathbb{R}$ be a deterministic square integrable function (w.r.t. Lebesgue measure, i.e. $\int_0^T f(t)\,\mathrm{d}t < \infty$). Show that the stochastic process $X_t := \int_0^t f(s)\,\mathrm{d}W_s$ has independent increments.

3. Let W_t be a Wiener process and let $f : [0,T] \to \mathbb{R}$ be a deterministic square integrable function (w.r.t. Lebesgue measure, i.e. $\int_0^T f(t)\,\mathrm{d}t < \infty$). Show that the stochastic process $X_t := \int_0^t f(s)\,\mathrm{d}W_s$ is a Gaussian process (meaning that the finite dimensional distributions are multivariate Gaussian). *(Hint: this follows from the statements of the previous two exercises.)*

4. Let W_t be a Wiener process and for $t \in [0,1)$ let $Z_t = (1-t)\int_0^t \frac{1}{1-s}\,\mathrm{d}W_s$. (We saw in class that this solves the stochastic differential equation $\mathrm{d}Z_t = -\frac{Z_t}{1-t}\,\mathrm{d}t + \mathrm{d}W_t$.) Also, for $t \in [0,1)$ let $X_t = W_t - tW_1$ be the Brownian bridge process. Show that Z and X have the same law. *(Hint: show that both are Gaussian processes and that they have the same expectations and covariances.)*

5. Let W_t be a Wiener process. At time 0 George and Bill make a bet: If W reaches 1 before reaching -1, George wins a penny from Bill, but if W reaches -1 before reaching 1, then Bill wins a penny from George. No money moves until either 1 or -1 is reached. Due to the symmetry of the Wiener process, this is clearly a fair bet. Let X_t denote the winnings of George at time t, so
$$X_t = W_\tau \mathbf{1}_{\tau \geq t},$$
where the stopping time $\tau := \inf\{t : W_t \in \{-1,1\}\}$ is the first hitting time of either -1 or 1 and $\mathbf{1}$ denotes indicator function.

   Now, the Martingale Representation Theorem says that every L^2 martingale starting from 0, adapted to (the natural filtration of) W_t can be written as a stochastic integral $\int_0^t \Phi_t\,\mathrm{d}W_t$. But such an integral process was defined to have continuous paths, while X_t has a jump at τ.

   What's wrong?

6. Let W_t be a Wiener process and let $N_t = W_t^3 - 3tW_t$. Show that N_t is a martingale. *(Hint: use the Itô formula to write N_t as an Itô integral.)*

7. Let W_t be a Wiener process and let $X_t = \frac{1}{1+W_t}$. Find a stochastic differential equation (of the form $\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}W_t$) that X_t satisfies.

8. According to the Martingale Representation Theorem, any L^2 martingale M_t adapted to the (natural filtration of the) Wiener process W_t can be written uniquely as $\mathrm{d}M_t = \phi_t\,\mathrm{d}W_t$ with some progressively measurable process ϕ_t. Define the process $V_t := \int_0^t \phi^2(s)\,\mathrm{d}s$ and call it the *quadratic variation process of M_t*.

   a.) Show that $(M_t)^2 - V_t$ is a martingale.

b.) What is the quadratic variation process of W_t?

c.) What is the quadratic variation process of $(W_t)^2 - t$?