Stochastic Differential Equations
Problem Set 1
Brownian Motion: Construction and Basic Properties

1.1 Let

$$\varphi : \mathbb{R} \to \mathbb{R}_+, \qquad \varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \text{be the standard normal density function,}$$

$$\Phi : \mathbb{R} \to [0,1], \quad \Phi(x) := \int_{-\infty}^{x} \varphi(y)dy, \quad \text{be the standard normal distribution function.}$$

Prove that for any $x > 0$

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\varphi(x) < 1 - \Phi(x) < \frac{1}{x}\varphi(x).$$

Hint: Compare the derivatives.

1.2 For every $n \in \mathbb{N}$ let $X_1^{(n)}, X_2^{(n)}, \dots, X_n^{(n)}$ be i.i.d. normal random variables with

$$\mathbf{E}\left(X_j^{(n)}\right) = 0, \qquad \mathbf{Var}\big(X_j^{(n)}\big) = \frac{1}{n}, \qquad j = 1, \dots, n.$$

Define the stochastic process $t \mapsto B^{(n)}(t)$, $t \in [0,1]$ as follows:

$$B^{(n)}(t) := \sum_{j=1}^{\lfloor nt \rfloor} X_j^{(n)}.$$

(a) Compute the expectations and covariances

$$\mathbf{E}\left(B^{(n)}(t)\right) =?, \qquad \mathbf{Cov}\big(B^{(n)}(t), B^{(n)}(s)\big) =?, \qquad s, t \in [0,1],$$

and their limits as $n \to \infty$.

(b) What is the joint distribution of the random variables $\{B^{(n)}(t) : t \in [0,1]\}$?

(c) Let

$$\delta_n := \max\left\{ \left| B^{(n)}(t+) - B^{(n)}(t-) \right| : t \in [0,1] \right\}.$$

(In plain words: δ_n is the largest jump discontinuity of the process $\{B^{(n)}(t) : t \in [0,1]\}$.)
Prove that for any fixed $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left(\delta_n \geq \varepsilon\right) = 0.$$

Hint: Note that $\delta_n = \max_{1\leq j\leq n} \left| X_j^{(n)} \right|$ and use the upper bound from problem 1.1.

1.3 For every $n \in \mathbb{N}$ let $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$ be i.i.d. Poisson random variables with parameter $1/n$. So,

$$\mathbf{E}\left(Y_j^{(n)}\right) = \frac{1}{n}, \qquad \mathbf{Var}\big(Y_j^{(n)}\big) = \frac{1}{n}, \qquad j = 1, \ldots, n.$$

Define the stochastic process $t \mapsto B^{(n)}(t)$, $t \in [0,1]$ as follows:

$$Z^{(n)}(t) := \sum_{j=1}^{\lfloor nt \rfloor} \left(Y_j^{(n)} - \frac{1}{n} \right).$$

(a) Compute the expectations and covariances

$$\mathbf{E}\left(Z^{(n)}(t)\right) = ?, \qquad \mathbf{Cov}\big(Z^{(n)}(t), Z^{(n)}(s)\big) = ?, \qquad s, t \in [0,1],$$

and their limits as $n \to \infty$.

(b) What is the joint distribution of the random variables $\{Z^{(n)}(t) : t \in [0,1]\}$? Explain in plain words.

(c) Let

$$\delta_n := \max\left\{ \left| Z^{(n)}(t+) - Z^{(n)}(t-) \right| : t \in [0,1] \right\}.$$

(In plain words: δ_n is the largest jump discontinuity of the process $\{Z^{(n)}(t) : t \in [0,1]\}$.)
Compute, for $\varepsilon > 0$ fixed,

$$\lim_{n\to\infty} \mathbf{P}\left(\delta_n \geq \varepsilon\right).$$

Hint: Note that $\delta_n = \max_{1\leq j\leq n} \left| Y_j^{(n)} \right|$ and use all you know about Poisson random variables.

1.4 Interpret the results of problems 1.2, respectively, 1.3.

1.5 (a) Let $Y_1, Y_2, \dots, Y_n$ be random variables with $\mathbf{E}(Y_j) = 0$ and $\mathbf{Cov}(Y_i, Y_j) =: c_{i,j}$. Assume that the covariance matrix $C := (c_{i,j})_{i,j=1}^n$ is non-degenerate, $\det(C) \neq 0$. Prove that the random variables $Y_1, Y_2, \dots, Y_n$ are *jointly Gaussian* if and only if there exist i.i.d. $\mathcal{N}(0,1)$-distributed random variables $X_1, X_2, \dots, X_n$ and real coefficients $(a_{i,j})_{i,j=1}^n$ such that

$$Y_i = \sum_{j=1}^n a_{ij} X_j.$$

Hint: Express the matrix $A = (a_{i,j})_{i,j=1}^n$ from the covariance matrix $C = (c_{i,j})_{i,j=1}^n$.

(b) Let $t \mapsto B(t)$ be standard $1d$ Brownian motion and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$. Explain why it follows from the definition of Brownian motion (i.e. independent and Gaussian increments) that the random variables $B(t_1), B(t_2), \dots, B(t_n)$ have jointly Gaussian distribution.

(c) Determine the covariance matrix of the random variables $B(t_1), B(t_2), \dots, B(t_n)$.

1.6 Let $t \mapsto B(t)$ be standard $1d$ Brownian motion. Prove that the following processes are also standard $1d$ Brownian motions:

(a) The rescaled process: $X(t) := a^{-1/2} B(at)$, where $a > 0$ is fixed parameter.

(b) The time reversed process: $Y(t) := tB(1/t)$.

(c) The backwards process: $Z(t) := B(T) - B(T-t)$, where $T > 0$ is fixed and $t \in [0, T]$.

Hint: Prove that the processes $X(t), Y(t), Z(t)$ are Gaussian and compute their covariances.

1.7 For $j = 1, \dots, n$, let $t \mapsto B_j(t)$, be independent $1d$ Brownian motions with variance σ_j^2, and a_j fixed real numbers. Prove that the process $t \mapsto Z(t) := \sum_{j=1}^n a_j B_j(t)$ is also a $1d$ Brownian motion. Determine the variance of the process $Z(t)$.

1.8 Let $t \mapsto B(t)$ be standard $1d$ Brownian motion. Determine (without painful computations) the conditional probability

$$\mathbf{P}\big(B(2) > 0 \mid B(1) > 0\big).$$

1.9 Show that $1d$ Brownian motion changes sign infinitely many times in any time interval $[0, \delta]$ of positive length δ.

1.10 Let $t_* \in [0,1]$ be arbitrary but fixed. Let $\frac{1}{2} < \alpha \leq 1$. Show that $B(t)$ is almost surely *not* α-Hölder continuous at t_*, meaning that there are no $\delta > 0$ and $C < \infty$ such that $|B(t_* + h) - B(t_*)| \leq C|h|^\alpha$ whenever $|h| \leq \delta$. *(Hint: look at the proof of non-differentiability at deterministic t – or just calculate the probablity of* $|B(t_* + h) - B(t_*)| \leq C|h|^\alpha$ *for a given h.)*

Solution: Let $A_{C,\delta}$ denote the event that $|B(t_* + h) - B(t_*)| \leq C|h|^\alpha$ for every h with $|h| \leq \delta$. First, let $C < \infty$ and $\delta > 0$ be fixed. $B(t_* + h) - B(t_*) \sim \mathcal{N}(0, h)$, so $B(t_* + h) - B(t_*) = \sqrt{h}\xi$ where $\xi \sim \mathcal{N}(0,1)$. So

$$\mathbf{P}\left(|B(t_* + h) - B(t_*)| \leq C|h|^\alpha\right) = \mathbf{P}\left(|\xi| \leq C|h|^{\alpha - \frac{1}{2}}\right) \leq \frac{1}{\sqrt{2\pi}} 2C|h|^{\alpha - \frac{1}{2}}.$$

since $\alpha - \frac{1}{2} > 0$, this goes to 0 as $h \to 0$, so $\mathbf{P}\left(A_{C,\delta}\right) = 0$. Now let $C_n = n$ and $\delta_n = \frac{1}{n}$. Then

$$\mathbf{P}\left(\{B(t) \text{ is } \alpha\text{-Hölder continuous at } t_*\}\right) = \mathbf{P}\left(\bigcup_{n=1}^{\infty} A_{C_n,\delta_n}\right) \leq \sum_{n=1}^{\infty} \mathbf{P}\left(A_{C_n,\delta_n}\right) = 0.$$

1.11 Show that almost surely there is no point $t \in [0,1]$ where B is $\frac{2}{3}$-Hölder continuous. *(Hint: mimic the proof of nowhere-differentiability.)*

1.12 *(Based on Exercise 8.1.3. from [1].)* Let $B(t)$ be a standard Brownian motion (Wiener process). Fix $t > 0$ and for $n = 0, 1, 2, \ldots$ let

$$V_n = \sum_{m=0}^{2^n - 1} \left(B\left(\frac{m+1}{2^n} t\right) - B\left(\frac{m}{2^n} t\right) \right)^2.$$

Calculate the expectation and the variance of V_n. Use the Borel-Cantelli lemma to show that $V_n \to t$ almost surely as $n \to \infty$.

Solution: For a given n, the squared increments $X_{n,m} := \left(B\left(\frac{m+1}{2^n}t\right) - B\left(\frac{m}{2^n}t\right)\right)^2$ are independent and identically distributed (for $m = 0, 1, \ldots, 2^n - 1$), with $X_{n,m} \sim \left(\sqrt{\frac{t}{2^n}}\xi\right)^2 = \frac{t}{2^n}\xi^2$, where $\xi \sim \mathcal{N}(0,1)$, meaning that $\mathbf{E}\left(X_{n,m}\right) = \frac{t}{2^n}\mathbf{E}\left(\xi^2\right) = \frac{t}{2^n}$ and $\mathbf{Var}\big(X_{n,m}\big) = \left(\frac{t}{2^n}\right)^2 \mathbf{Var}\big(\xi^2\big) = \frac{const}{4^n}$. This means that $\mathbf{E}\left(V_n\right) = \sum_{m=0}^{2^n-1} \mathbf{E}\left(X_{n,m}\right) = t$ and $\mathbf{Var}\big(V_n\big) = \sum_{m=0}^{2^n-1} \mathbf{Var}\big(X_{n,m}\big) = \frac{const}{2^n}$. Now Chebyshev's inequality says that

$$\mathbf{P}\left(|V_n - t| \geq \frac{1}{n}\right) \leq \frac{\mathbf{Var}\big(V_n\big)}{\left(\frac{1}{n}\right)^2} = const\frac{n^2}{2^n},$$

which is summable. Now the first Borell-Cantelli lemma implies that almost surely $|V_n - t| < \frac{1}{n}$ for all but finitely many n, so $V_n \to t$.

1.13 For $\alpha \geq 0$ Let $m_\alpha = \mathbf{E}(|\xi|^\alpha)$ and $c_\alpha = \mathbf{Var}\big(|\xi|^\alpha\big)$, where ξ is standard Gaussian. Express c_α using m_α and $m_{2\alpha}$.

1.14 Let $X_1, X_2, \ldots$ be random variables such that $\mathbf{E}(X_n) \to \infty$ and $\frac{\mathbf{Var}(X_n)}{(\mathbf{E}(X_n))^2} \to 0$ as $n \to \infty$. Show that $X_n \to \infty$ in probability – that is: $\mathbf{P}(X_n \leq M) \to 0$ for any $M < \infty$.

Solution: We just use Chebyshev's inequality. For any fixed M we have $M < \mathbf{E}(X_n)$ for large enough n, so

$$\begin{aligned}\mathbf{P}(X_n \leq M) &\leq \mathbf{P}(|X_n - \mathbf{E}(X_n)| \geq \mathbf{E}(X_n) - M) \\ &\leq \frac{\mathbf{Var}(X_n)}{(\mathbf{E}(X_n) - M)^2} = \frac{\mathbf{Var}(X_n)}{(\mathbf{E}(X_n))^2}\left(\frac{\mathbf{E}(X_n)}{\mathbf{E}(X_n) - M}\right)^2 \to 0.\end{aligned}$$

1.15 Find eigenvalues and eigenvectors of $K : L^2([0,1]) \to L^2([0,1])$ where $(Kf)(t) = \int_0^1 \mathcal{K}(t,s)f(s)\mathrm{d}s$ and $\mathcal{K}(t,s) = \min\{t,s\}$. *(Hint 1: the solution is given in the next exercise. If you are tough, don't look at it. Checking the solution is much easier than finding it. Hint 2: try $f(s) = e^{\lambda s}$ first. It will not work, but you will see how to fix it.)*

1.16 On the Hilbert space $\mathcal{L}^2([0,1], dx)$ define the self-adjoint compact (actually: Hilbert-Schmidt) operator

$$Kf(t) := \int_0^1 \min\{t,s\} f(s) ds.$$

Prove that

$$\lambda_n = \frac{4}{\pi^2(2n-1)^2}, \qquad \psi_n(t) = \sqrt{2}\sin\left(\frac{\pi(2n-1)}{2}t\right), \qquad n = 1, 2, \ldots$$

are eigenvalues and eigenvectors of the operator K.

1.17 Check that for $t, s \in [0,1]$

$$\min\{t,s\} = \sum_{n=1}^{\infty} \lambda_n \psi_n(t)\psi_n(s)$$

where

$$\lambda_n = \frac{4}{\pi^2(2n-1)^2}, \qquad \psi_n(t) = \sqrt{2}\sin\left(\frac{\pi(2n-1)}{2}t\right), \qquad n = 1, 2, \ldots$$

(Hint: fix $t \in [0,1]$, and look at both sides of the equation as a function of s. Then the RHS is the Fourier series of a function on $\mathbb{R}$ which is periodic with some period l (and it happens that $l \neq 1$). This function is odd. So extend the LHS from $[0,1]$ to $\mathbb{R}$ to get an odd, l-periodic and continuous *function to make sure that its equal to its Fourier series poitwise. Now just calculate the Fourier expansion.)*

1.18 For $n = 1, 2, \ldots$ let $c_n = \frac{2}{\pi(2n-1)}$ and $\psi_n(t) = \sqrt{2}\sin\left(\frac{\pi(2n-1)}{2}t\right)$. Let $\xi_1, \xi_2, \ldots$ be independent standard Gaussian random variables.

a.) Prove that the series

$$B(t) = \sum_{n=1}^{\infty} c_n \xi_n \psi_n(t)$$

is almost surely convergent for every fixed $t \in [0,1]$. *(Hint 1 (overshooting): the Kolmogorov three series theorem can be applied. Hint 2: The partial sum is a martingale. Apply a martingale convergence theorem.)*

Solution: Fix t and let Let $X_N = \sum_{n=1}^{N} c_n \xi_n \psi_n(t)$. This is a sum of independent random variables with zero expectation, so it's clearly a martingale. Since $|\psi_n(t)| \leq \sqrt{2}$ for all n,

$$\mathbf{Var}\big(X_n\big) = \sum_{n=1}^{N} \mathbf{Var}\big(c_n \xi_n \psi_n(t)\big) \leq \sum_{n=1}^{N} c_n^2 \sqrt{2}^2 \mathbf{Var}\big(\xi_n\big) = 2\sum_{n=1}^{\infty} c_n^2 =: M < \infty,$$

so the L^2 martingale convergence theorem implies that X_N converges almost surely (and also in L^2).

Alternatively: X_N^2 is a nonnegative submartingale with bounded expectation, so the martingale convergence theorem implies that it converges almost surely.

b.) Prove that the series

$$B = \sum_{n=1}^{\infty} c_n \xi_n \psi_n$$

is convergent in $L^2([0,1])$.

Solution: SORRY, this exercise was not formulated precisely. Actually, it should have been "Consider $X_N := \sum_{n=1}^{N} c_n \xi_n \psi_n$ as a random element of $L^2([0,1])$ (since it is a random function of t). Show that this sequence is almost surely convergent in $L^2([0,1])$."

So the solution: since $\psi_1, \psi_2, \ldots$ are orthonormal in $L^2([0,1])$, the convergence of X_N is equivalent to the convergence of the sum $\sum_{n=1}^{\infty} c_n^2 \xi_n^2$ (see the footnote[1]). Now $\sum_{n=1}^{\infty} c_n^2 \xi_n^2 < \infty$ almost surely, since it's a sum of nonnegative random variables, and even its expectation is finite:

$$\mathbf{E}\left(\sum_{n=1}^{\infty} c_n^2 \xi_n^2\right) = \sum_{n=1}^{\infty} c_n^2 \mathbf{E}\left(\xi_n^2\right) = \sum_{n=1}^{\infty} c_n^2 < \infty.$$

1.19 Show that the function

$$\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \qquad \phi(t,x) := \frac{1}{\sqrt{t}} \varphi\left(\frac{x}{\sqrt{t}}\right)$$

solves the heat equation

$$\partial_t \phi(t,x) = \frac{1}{2} \partial_x^2 \phi(t,x).$$

1.20 Exercise 1 implies that if ξ is a standard Gaussian random variable and $x \geq 1$, then

$$\mathbf{P}\left(|X| \geq x\right) \leq \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}.$$

Use this to show that if $\xi_1, \xi_2, \ldots$ are i.i.d. standard Gaussian, then, with probability 1, the event $\{|\xi_n| > 2 \ln n\}$ occurs for at most finitely many n-s.

1.21 Fix $t > 0$ and let $Y \sim \mathcal{N}(0,t)$. Let $\xi \sim \mathcal{N}(0, \sigma^2)$ with some $\sigma > 0$ be independent of Y and let $X = \frac{Y}{2} + \xi$.

a.) How should σ be chosen for X and $Y - X$ to be independent?

b.) In this case, what is the variance of X?

1.22 *Paul Lévy's construction of the Wiener process.* In a possible construction of the Wiener process (or Brownian motion) on $[0,1]$ we define a sequence of piecewise linear continuous random functions so that we first define f_n at dyadic rationals that are multiples of $\frac{1}{2^n}$, inheriting every second value (at multiples of $\frac{1}{2^{n-1}}$) form f_{n-1}, and setting the values at the remaining points (of the form $\frac{2k-1}{2^n}$) to be the average of the two neighbouring values, plus an independent Gaussian random value with mean 0 and variance $\frac{1}{2^{n+1}}$. Then we extend f_n to $[0,1]$ piecewise linearly.

Formally: we take independent standard Gaussian random variables ξ_0 and $\xi_{n,k}$ where $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2^{n-1}$. Then

[1] Indeed, for $M > N$ we have $||X_M - X_N||^2 = \sum_{n=N+1}^{M} c_n^2 \xi_n^2$, so X_N is Cauchy in L^2 if and only if $\sum_{n=N+1}^{M} c_n^2 \xi_n^2 \to 0$ as $N \to \infty$.

- In the 0th step we fix $f_0(0) = 0$ and $f_0(1) = \xi_0$. We connect these two values linearly.
- In the 1st step we leave $f_1(0) = f_0(0)$ and $f_1(1) = f_0(1)$, but also set $f_1(\frac{1}{2}) = f_0(\frac{1}{2}) + \frac{1}{2}\xi_{1,1}$. We connect these three values linearly.
- ...in the nth step we leave $f_n\left(\frac{k}{2^{n-1}}\right) = f_{n-1}\left((\frac{k}{2^{n-1}}\right)$ for $k = 0, 1, \ldots, 2^{n-1}$, but also set $f_n\left(\frac{k-\frac{1}{2}}{2^{n-1}}\right) = f_{n-1}\left(\frac{k-\frac{1}{2}}{2^{n-1}}\right) + \frac{1}{\sqrt{2^{n+1}}}\xi_{n,k}$ for $k = 1, \ldots, 2^{n-1}$. We connect these $2^n + 1$ values linearly.

Notice that, in this construction, the difference $g_n := f_{n+1} - f_n$ is the sum of 2^n "tent" maps with disjoint supports and i.i.d. Gaussian "heights".

(a) Use the statement of Exercise 20 to show that, with probability 1, the series

$$\lim_{n\to\infty} f_n = f_0 + \sum_{n=0}^{\infty} g_n$$

is uniformly absolutely convergent.

(b) Check that the limit is a Wiener process.

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)