Stochastic Differential Equations
Problem Set 4
Stochastic differential equations, Dynkin formula, Girsanov theorem

4.1 Check that the following processes solve the corresponding SDE's, where $B(t)$ is 1-dimensional standard Brownian motion:

(a) $X(t) = e^{B(t)}$, with $B(0) = b$ solves

$$dX(t) = \frac{1}{2}X(t)dt + X(t)dB(t), \qquad X(0) = e^b.$$

(b) $X(t) = \frac{B(t)}{1+t}$, with $B_0 = b$, solves

$$dX(t) = -\frac{X(t)}{1+t}dt + \frac{1}{1+t}dB_t, \qquad X(0) = b.$$

(c) $X(t) = \sin B(t)$, with $B(0) = b \in (-\pi/2, \pi/2)$, and $t < \min\{t : |\,B(t)\,| = \pi/2\}$, solves

$$dX(t) = -\frac{1}{2}X(t)dt + \sqrt{1 - X(t)^2}dB_t, \qquad X(0) = \sin b.$$

(d) $\big(X_1(t), X_2(t)\big) = \big(\cosh B(t), \sinh B(t)\big)$, with $B(0) - b$, solves

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dt + \begin{pmatrix} X_2(t) \\ X_1(t) \end{pmatrix} dB(t).$$

4.2 Let $B(t)$ be a standard 1-dimensional Brownian motion with $B(0) = b$, and $\big(U(t), V(t)\big) := \big(\cos B(t), \sin B(t)\big)$. Write down in vectorial notation the SDE driving the 2-dimensional process $\big(U(t), V(t)$.

4.3 Solve the following SDE's, where $B(t)$ is 1-dimensional standard Brownian motion starting from $B(0) = 0$:

(a)

$$dX(t) = -X(t)dt + e^{-t}dB(t).$$

(b)

$$dX(t) = rdt + \alpha X(t)dB(t),$$

with $r, \alpha \in \mathbb{R}$ constants.
Hint: Multiply by $\exp\big(-\alpha B(t) + \frac{\alpha^2}{2}t\big)$.

(c) Now, $X(t) = \big(X_1(t), X_2(t)\big) \in \mathbb{R}^2$, and $B(t) = \big(B_1(t), B_2(t)\big)$ is standard 2-dimensional Brownian motion.

$$\begin{aligned} dX_1(t) &= X_2(t)dt + \alpha dB_1(t) \\ dX_2(t) &= -X_1(t)dt + \beta dB_2(t), \end{aligned}$$

or in vector notation,

$$dX(t) = JX(t)dt + AdB(t), \qquad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Hint: Multiply by left by e^{-Jt}.

4.4 The Ornstein-Uhlenbeck process:

(a) Solve explicitly the stochastic differential equation

$$dX(t) = -\gamma X(t)dt + adB(t), \qquad X(0) = x_0,$$

and show that the process $X(t)$ is Gaussian.
Hint: Multiply by $e^{\gamma t}$.

(b) Compute $\mathbf{E}\,(X(t))$ and $\mathbf{Cov}\big(X(s), X(t)\big)$.

(c) Let $Y_k^{(n)}$ be the Markov chain on the state space $S^{(n)} := \{0, 1, \dots, n\}$ with transition matrix

$$P_{i,j}^{(n)} = \frac{i}{n}\delta_{i-1,j} + \frac{n-i}{n}\delta_{i+1,j}, \qquad i, j \in S^{(n)}.$$

The Markov chain $Y_k^{(n)}$ is called *Ehrenfest's Urn Model* (or *Dogs and Fleas*). Define the sequence of continuous time processes

$$X^{(n)}(t) := \frac{Y_{\lfloor nt \rfloor}^{(n)} - (n/2)}{\sqrt{n}}, \qquad t \geq 0.$$

Write down an *approximate stochastic differential equation* for $X^{(n)}(t)$, with time increments $dt = \frac{1}{n}$ and conclude (non-rigorously) that the Ornstein-Uhlenbeck process is – in some sense – the limit of the processes $X^{(n)}(t)$ (that is: the *scaling limit* of Ehrenfest's Urn Model.)

4.5 Write down the infinitesimal generator as elliptic differential operator for the following Itô diffusions:

(a) $dX(t) = \beta dt + \alpha X(t) dB(t)$.

(b) $dY(t) = \begin{pmatrix} dt \\ dX(t) \end{pmatrix}$, where $dX(t) = -\gamma X(t) dt + \alpha dB(t)$.

(c) $\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ X_2(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ e^{X_1(t)} \end{pmatrix} dB(t)$.

(d) $\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & X_1 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}$.

4.6 Find an Itô diffusion (i.e., write down the SDE for it) whose infinitesimal generator is the following:

(a) $Af(x) = f'(x) + f''(x)$, $f \in C_0^2(\mathbb{R})$.

(b) $Af(t,x) = \dfrac{\partial f}{\partial t} + cx\dfrac{\partial f}{\partial x} + \dfrac{1}{2}\alpha^2 x^2 \dfrac{\partial^2 f}{\partial x^2}$, $f \in C_0^2(\mathbb{R}^2)$.

4.7 Let $X(t)$ be a geometric Brownian motion, i.e. strong solution of the following SDE

$$dX(t) = \beta X(t) dt + \alpha X(t) dB(t), \qquad X_0 = x > 0,$$

where $\alpha > 0$, $\beta \in \mathbb{R}$ are fixed parameters.

(a) Find the generator A of the diffusion $t \mapsto X(t)$ and compute $Af(x)$ when $f : \mathbb{R}_+ \to \mathbb{R}$ is $f(x) = x^\gamma$, γ constant.

(b) Let $0 < r < R < \infty$, and $r \le x \le R$. using Dynkin's formula, compute

$$\mathbf{P}\big(\tau_r < \tau_R \bigm| X(0) = x\big),$$

where τ_r, and τ_R are the first hitting times of r, respectively, R.
Hint: Solve the boundary value problem $Af(x) = 0$ for $r < x < R$, with $f(r) = 1$, $f(R) = 0$.

(c) Assume $\beta < \alpha^2/2$. What is $\mathbf{P}\big(X(t) \text{ ever hits } R \bigm| X(0) = x\big)$?

(d) Assume $\beta > \alpha^2/2$. What is $\mathbf{P}\big(X(t) \text{ ever hits } r \mid X(0) = x\big)$?

4.8 (a) Find the generator of the δ-dimensional Bessel process, $BES(\delta)$

$$dY^{(\delta)}(t) = \frac{\delta - 1}{2Y^{(\delta)}(t)} dt + dB(t)$$

on $\mathbb{R}_+$.

(b) Let $0 < r < R < \infty$, and $r \le x \le R$. using Dynkin's formula, compute

$$\mathbf{P}\big(\tau_r < \tau_R \mid Y^{(\delta)}(0) = x\big),$$

where τ_r, and τ_R are the first hitting times of r, respectively, R.
Hint: Solve the boundary value problem $Af(x) = 0$ for $r < x < R$, with $f(r) = 1$, $f(R) = 0$. Note that the solutions are *qualitatively different* for $\delta \in [0, 2)$, $\delta = 2$, respectively, $\delta > 2$.

(c) Show that $BES(\delta)$ is transient if $\delta > 2$.

(d) Show that $BES(2)$ almost surely hits all points in $(0, \infty)$, but never hits 0.

(e) Show that for $\delta \in [0, 2)$ $BES(\delta)$ almost surely hits 0 (no matter where it starts).

4.9 Show that the solution $u(t, x)$ of the initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t}(t, x) = \frac{1}{2}\beta^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) + \alpha x \frac{\partial u}{\partial x}(t, x), \qquad t > 0,\ x \in \mathbb{R}, \\ &u(0, x) = f(x), \qquad (f \in C_K^2(\mathbb{R}) \text{ given}) \end{aligned}$$

can be expressed as follows:

$$\begin{aligned} u(t, x) &= \mathbf{E}\left(f\big(x \exp\{\beta B(t) + (\alpha - \beta^2/2)t\}\big)\right) \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f\big(x \exp\{\beta y + (\alpha - \beta^2/2)t\}\big) \exp(-y^2/(2t)) dy, \qquad t > 0\,. \end{aligned}$$

In this expression $t \mapsto B(t)$ is standard 1-dimensional Brownian motion with $B(0) = 0$.

4.10 *[Change of conditional expectation]*
Let $\mathbf{Q}$ and $\mathbf{P}$ be two probability measures on $(\Omega, \mathcal{F})$, with $\mathbf{Q} \ll \mathbf{P}$, and Radon-Nikodym derivative $\frac{d\mathbf{Q}}{d\mathbf{P}}(\omega) = \varrho(\omega)$. Let $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra. Show that, for any $\mathcal{F}$-measurable random variable X, we have

$$\mathbf{E}_{\mathbf{Q}}\big(X \mid \mathcal{G}\big) = \frac{\mathbf{E}_{\mathbf{P}}\big(\varrho X \mid \mathcal{G}\big)}{\mathbf{E}_{\mathbf{P}}\big(\varrho \mid \mathcal{G}\big)} \tag{1}$$

4.11 *[A discrete version of Girsanov's formula]*

Let $\Omega_n := \{\mathsf{H}, \mathsf{T}\}^n$, $\mathbf{P}$ be the probability measure on Ω_n given by tossing a biased coin n times independently which gives probability $2/3$ to H, and $\mathbf{Q}$ the probability measure given by tossing a fair coin n times independently. Let $Z_n(\omega) := \frac{d\mathbf{Q}}{d\mathbf{P}}(\omega)$, and consider the martingale (with respect to the measure $\mathbf{P}$) $Z_m := \mathbf{E}_{\mathbf{P}}\big(Z_n \,\big|\, \mathcal{F}_m \big)$ for $m \leq n$.

(a) Give *explicitly* the distribution of Z_{m+1} given $Z_m, \ldots, Z_1$.

(b) Note that (1) of the previous exercise translates to $\mathbf{E}_{\mathbf{Q}}\big(X \,\big|\, \mathcal{F}_m \big) = (Z_m)^{-1}\mathbf{E}_{\mathbf{P}}\big(X Z_n \,\big|\, \mathcal{F}_m \big)$. Check this numerically for $n = 3$, $m = 2$, $X = \#\{\text{heads in } (\omega_1, \omega_2, \omega_3)\}$.

(c) Interpret this exercise as a discrete version of Girsanov's theorem.

4.12 *[Cameron-Martin theorem]*

(a) Let $f \in L^2[0,1]$ be a deterministic function and $F(t) := \int_0^t f(u)du$, $t \in [0,1]$. Show that, if $t \mapsto B(t)$ is standard 1d Brownian motion, then the laws of the processes $\{t \mapsto F(t) + B(t) : t \in [0,1]\}$ and $\{t \mapsto B(t) : t \in [0,1]\}$ are mutually absolutely continuous w.r.t. each other. Compute the Radon-Nikodym derivatives.

(b) If $F(t)$ is such that the above $f(t)$ does not exist, then the laws of the two processes are mutually singular.

4.13 Let $B(t) = (B_1(t), B_2(t))$, $t \leq T$, be a 2-dimensional standard Brownian motion on the probability space $(\Omega, \mathcal{F}_T, \mathbf{P})$. Find a probability measure $\mathbf{Q}$ on $\mathcal{F}_T$ that is mutually absolutely continuous w.r.t. $\mathbf{P}$, and under which the following process $t \mapsto Y(t)$ becomes a martingale:

(a)

$$dY(t) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} dt + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}, \qquad t \leq T.$$

(b)

$$dY(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}, \qquad t \leq T.$$

4.14 Let $B(t)$ be standard 1-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $Y(t) = t + B(t)$. For each $T > 0$, find $\mathbf{Q}_T \sim \mathbf{P}$ on $\mathbb{F}_T$ such that $\{t \mapsto Y(t)\}_{t \leq T}$ becomes a Brownian motion under $\mathbb{Q}_T$.

(a) Show that there exists a probability measure $\mathbf{Q}$ on $\mathcal{F}$ such that $\mathbf{Q}|_{\mathcal{F}_T} = \mathbb{Q}_T$ for all $T > 0$.

(b) Show that $\mathbf{P}\big(\lim_{t\to\infty} Y(t) = \infty\big) = 1$, while $\mathbf{Q}\big(\lim_{t\to\infty} Y(t) = \infty\big) = 0$. Why does not this contradict Girsanov's theorem?

4.15 Let $b : \mathbb{R} \to \mathbb{R}$ be Lipschitz, and $t \mapsto X(t)$ be the unique strong solution of the 1-dimensional SDE

$$dX(t) = b(X(t))dt + dB(t), \qquad X(0) = x \in \mathbb{R}.$$

(a) Use Girsanov's theorem to prove that for any $M < \infty$, $x \in \mathbb{R}$, and $t > 0$, we have $\mathbf{P}\left(X(t) > M\right) > 0$.

(b) Choose $b(x) = -r$, where $r > 0$ is a constant. Prove that, for all x, we have $\lim_{t\to\infty} X(t) = -\infty$, a.s. Compare this fact with the result in part (a).