(1) $\mathbb{P}\left(\sup_{t\in[0,T]} |\mathcal{J}_{n+1}(t)-\mathcal{J}_n(t)|>\frac{1}{2^n}\right)=\mathbb{P}\left(\sup_{t\in[0,T]} M_n(t)>\frac{1}{4^n}\right)$

where $M_n(t):=(\mathcal{J}_{n+1}(t)-\mathcal{J}_n(t))^2$ is a non-negative submartingale (because it's the square of a martingale).

So by Doob's maximal inequality

$$\mathbb{P}\left(\sup_{t\in[0,T]} M_n(t)>\frac{1}{4^n}\right)\le\frac{\mathbb{E}\,M_n(T)}{\frac{1}{4^n}}\underset{\text{the exercise}}{\overset{\text{assumption of}}{\le}}\frac{1/8^n}{1/4^n}=\frac{1}{2^n}.$$

This is summable, so by the 1st Borel-Cantelli lemma,

$\sup_{t\in[0,T]}|\mathcal{J}_{n+1}(t)-\mathcal{J}_n(t)|\le\frac{1}{2^n}$ for all but finitely many n,

almost surely. This means that $\mathcal{J}_n$ is almost surely Cauchy in the sup norm. $\square$

2 Let $f(x) = \ln x$ and $Y(t) = f(X(t)) = \ln X(t)$.

Then $f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$, so by Itô's formula

$$dY(t) = \frac{1}{X(t)} dX(t) - \frac{1}{2X^2(t)} (dX(t))^2 = dt + dB(t) - \frac{\cancel{X^2(t)}dt}{2\cancel{X^2(t)}} = \frac{1}{2} dt + dB(t)$$

$\Rightarrow Y(t) = Y(0) + \frac{1}{2} t + B(t)$ where $Y(0) = \ln X(0)$

$$\Rightarrow \boxed{X(t) = e^{Y(t)} = X(0) e^{\frac{1}{2} t + B(t)}}$$

3 a.) $dX(t) = -X(t)dt + \sqrt{1-X^2(t)}\,dB(t) = b(X(t))\,dt + \sigma(X(t))\,dB(t)$

with $b(x) = -x$, $\sigma(x) = \sqrt{1-x^2}$, so $\frac{1}{2}\sigma^2(x) = \frac{1-x^2}{2}$

So the generator is

$$(Af)(x) = b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) = -xf'(x) + \frac{1-x^2}{2}f''(x).$$

b.) If $f(x) = \ln(1+x) - \ln(1-x)$, then

$$f'(x) = \frac{1}{1+x} + \frac{1}{1-x} = \frac{1-x+1+x}{1-x^2} = \frac{2}{1-x^2}$$

$$f''(x) = \frac{+2\cdot 2x}{(1-x^2)^2} = \frac{4x}{(1-x^2)^2}$$

so $(Af)(x) = -x\frac{2}{1-x^2} + \frac{\cancel{1-x^2}}{\cancel{2}}\frac{\overset{2}{\cancel{4}}x}{(1-x^2)^{\cancel{2}}} = 0$ ✓

c.) Let $\tau = \min\{\tau_a, \tau_b\}$. By Dynkin's formula

$$\mathbb{E}_{x_0} f(X(\tau)) = f(x_0) + \mathbb{E}\int_0^\tau (Af)(X(s))\,ds = f(x_0),$$

since $Af \equiv 0$.

Let $p_a = \mathbb{P}_{x_0}(\tau_a < \tau_b)$, $p_b = \mathbb{P}_{x_0}(\tau_b < \tau_a)$, so

$$\begin{cases} p_a f(a) + p_b f(b) = f(x_0) \\ p_a + p_b = 1 \end{cases} \iff p_a = \frac{f(b) - f(x_0)}{f(b) - f(a)}, \quad p_b = \frac{f(x_0) - f(a)}{f(b) - f(a)}$$

In particular, $$\boxed{p_a = \frac{\ln\frac{1+b}{1-b} - \ln\frac{1+x_0}{1-x_0}}{\ln\frac{1+b}{1-b} - \ln\frac{1+a}{1-a}}}$$

d.) $\mathbb{P}_{x_0}(\tau_a < \infty) = \lim_{b\nearrow 1} p_a(a, x_0, b) = 1.$

(4) The evolution of $p(t,x)$ is given by Kolmogorov's <u>forward</u> equation: in our case

$$\partial_t p(t,x) = \partial_x (x\, p(t,x)) + \frac{1}{2} \partial^2_{x^2} (1-x^2)\, p(t,x).$$

The function $p(t,x) := \begin{cases} \frac{1}{2}, & \text{if } -1 < x < 1,\ t \in \mathbb{R}^+ \\ 0, & \text{if } x \notin (-1,1) \end{cases}$

is clearly a solution:

$$\partial_x \left(x \cdot \frac{1}{2}\right) + \frac{1}{2} \partial^2_{x^2}\left((1-x^2)\frac{1}{2}\right) = \frac{1}{2}\partial_x\left[x - \frac{2x}{2}\right] = 0 \ ✓$$

This means that the uniform distribution remains unchanged in time. $\square$

[<u>Remark</u>: for a rigorous proof, one would need to check that non-differentiability at $x = \pm 1$ causes no problem, which is so because $+1$ and -1 are never reached by the process — which can be seen from the previous exercise.]