SDE exam, 13 June 2018.
Working time: 90 minutes

In the questions and exercises below, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ denotes a filtered probability space. Every stochastic process is *progressively measurable* w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$. $t \mapsto B(t)$ is a standard standard Brownian motion (in 1 or more dimensions, as it will always be clear from the context.)

1. 20 points

Let $T < \infty$ and

$$0 = s_{n,0} < s_{n,1} < \cdots < s_{n,n-1} < s_{n,n} = T$$

a sequence of partitions of the interval $[0, T]$, for which $\delta_n := \max_{0\leq j\leq n-1}(s_{n,j+1} - s_{n,j}) \to 0$ as $n \to \infty$. How much is the limit

$$\lim_{n\to\infty} \mathbf{E}\Big(\Big| \sum_{j=0}^{n-1} (B(s_{n,j+1}) - B(s_{n,j}))^2 \Big|^2 \Big) \quad ? \tag{1}$$

2. (a) **15 points**

Let $v : [0, \infty) \to \mathbb{R}$ be a process adapted to the filtration az $(\mathcal{F}_t)_{t\geq 0}$, the trajectory of which is almost surely a smooth (C^1) function. Prove directly (without reference to Itô calculus) that the process

$$t \mapsto X(t) := v(t)B(t) - \int_0^t v'(s)B(s)ds$$

is an $(\mathcal{F}_t)$-martingale.

(b) **15 points**

Using Itô's formula, write the process $t \mapsto X(t)$ in (2a) as an Itô integral.

3. Let $\delta \geq 0$ be a fixed parameter and let $t \mapsto Z_\delta(t) \in \mathbb{R}_+$ be the solution of the stochastic differential equation

$$dZ_\delta(t) = \frac{\delta}{2}dt + \sqrt{2Z(t)}dB(t), \qquad Z_\delta(0) = z > 0$$

on the time interval $t \in [0, \tau_0)$, where $\tau_0 := \inf\{s : Z_\delta(s) = 0\}$.

(a) **15 points**

Write the infinitesimal generator A of the diffusion process $t \mapsto Z_\delta(t)$, and the *general solution* of the differential equation $Af = 0$.

(b) **20 points**

Let $0 < a < z < b < \infty$ and

$$\tau_a := \inf\{s > 0 : Z_\delta(s) = a\}, \qquad \tau_b := \inf\{s > 0 : Z_\delta = b\}.$$

Using Dynkin's formula, calculate the probability $\mathbf{P}\big(\tau_a < \tau_b \bigm| Z_\delta(0) = z\big)$.

(c) **15 points**

Describe the asymptotic behaviour of the process $t \mapsto Z_\delta(t)$ in the $t \to \infty$ limit. Discuss its dependence on the parameter δ.