SDE makeup midterm solutions 2019.05.13, Imre Péter Tóth

(1) a) Let $\xi = B(1)$, $\eta = B(2)-B(1)$, so $\xi,\eta \sim N(0,1)$ and they are independent. Then

$$P(B(1)>0,\, B(2)>0) = P(\xi>0,\, \xi+\eta>0) = P((\xi,\eta)\in D) \text{ with}$$

Since the distribution of (ξ,η) is rotation symmetric, $P(B(1)>0, B(2)>0) = P((\xi,\eta)\in D) = \frac{3}{8}$

$$\Rightarrow P(B(1)>0 \mid B(2)>0) \overset{\text{def}}{=} \frac{P(B(1)>0,\, B(2)>0)}{P(B(2)>0)} = \frac{3/8}{1/2} = \underline{\underline{\frac{3}{4}}}$$

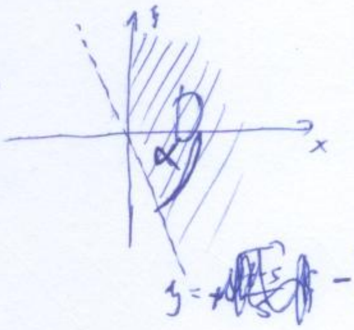
b) Similarly: let $B(2) = \sqrt{2}\,\xi$, $B(4)-B(2) = \sqrt{2}\,\eta$. Then $\xi,\eta \sim N(0,1)$ and they are independent. Again,

$$P(B(2)>0,\, B(4)>0) = P(\sqrt{2}\,\xi>0,\, \sqrt{2}\,\xi+\sqrt{2}\,\eta>0) = P(\xi>0,\, \xi+\eta>0) \overset{\text{previous}}{=\!=} \frac{3}{8} \Rightarrow P(B(4)>0 \mid B(2)>0) = \frac{3/8}{1/2} = \underline{\underline{\frac{3}{4}}}$$

c) Assume $s\le t$, let $B(s) = \sqrt{s}\,\xi$, $B(t)-B(s) = \sqrt{t-s}\,\xi$. Again, $\xi,\eta\sim N(0,1)$ and they are independent. Now

$$P(B(s)>0,\, B(t)>0) = P(\sqrt{s}\,\xi>0,\, \sqrt{s}\,\xi+\sqrt{t-s}\,\eta>0) = P\left(\xi>0,\, \eta>-\sqrt{\frac{s}{t-s}}\,\xi\right) =$$

$= P((\xi,\eta)\in D)$ with



Again, due to rotation symmetry,

$$P((\xi,\eta)\in D) = \frac{\pi/2+\alpha}{2\pi} = \frac{\pi/2 + \arctan\sqrt{\frac{s}{t-s}}}{2\pi}$$

Summarising: for any $s\ne t>0$ $P(B(s)>0,\, B(t)>0) = \frac{1}{4} + \frac{1}{2\pi}\arctan\sqrt{\frac{\min\{s,t\}}{|t-s|}}$

So $\boxed{P(B(s)>0 \mid B(t)>0) = \frac{1}{2} + \frac{1}{\pi}\arctan\sqrt{\frac{\min\{s,t\}}{|t-s|}}}$

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(1) ~~c.) continued~~

~~So~~ $\cancel{\mathbb{P}(B(s)>0, B(t)>0) = \frac{1}{4} + \overbrace{\frac{1}{4}\frac{2}{\pi}}^{\frac{1}{2\pi}} \arctan\sqrt{\frac{\min(s,t)}{|t-s|}}}$ ~~for any $t,s>0$~~

$$\cancel{\Rightarrow \boxed{\mathbb{P}(B(s)>0 \mid B(t)>0) = \frac{1}{2} + \frac{1}{\pi}\arctan\sqrt{\frac{\min(s,t)}{|t-s|}}}}$$

(2) a.) We know that $Y(t) := tB(\frac{1}{t})$ is a Wiener process so $X(t) = \frac{Y(t)}{t^{\beta}}$ with $\beta = 1-\alpha$

We know that $\frac{Y(t)}{t^{\beta}} \to 0$ a.s. for $\beta > \frac{1}{2}$

and is divergent for $\beta \le \frac{1}{2}$

(meaning $-\infty = \liminf\limits_{t\to\infty} \frac{Y(t)}{t^{\beta}} < \limsup\limits_{t\to\infty} \frac{Y(t)}{t^{\beta}} = \infty$ a.s.).

So $\boxed{X(t) \text{ is a.s. convergent} \iff \alpha < \frac{1}{2}}$

b.) $X(t) \sim N(0, (t^{\alpha})^2 \frac{1}{t}) = N(0, t^{2\alpha-1})$

If $2\alpha-1<0$, then $t^{2\alpha-1} \to 0$, so $X(t) \Rightarrow N(0,0) \equiv 0$

If $2\alpha-1=0$, then $t^{2\alpha-1}=1$, so $X(t) \sim N(0,1)$ for $\forall t$

If $2\alpha-1>0$, then $t^{2\alpha-1} \to \infty$, so $X(t)$ is divergent weakly.

So $\boxed{X(t) \text{ is weakly convergent} \iff \alpha \le \frac{1}{2}}$

(3) $X_n := \sum\limits_{k=1}^{n} \left[B(\frac{k}{n}) - B(\frac{k-1}{n})\right]^4$ is an approximation of the α-variation of $B(t)$ on $[0,1]$ with $\alpha = 4 > 2$,

so we know that it converges to 0 (strongly, weakly, in L^2 ...)

In particular, $X_n \Rightarrow 0$ means $\boxed{\lim\limits_{n\to\infty} \mathbb{P}(X_n > x) = 0 \text{ for every } x>0}$

4) a.) $\alpha=0$ will do: $B(t)$ is a martingale and $x \mapsto x^4$ is convex, so $X(t)=B^4(t)$ is a submartingale (Integrability is OK)

b.) $\alpha=1$ will also do: $Y(t):=B^2(t)-t$ is a martingale and $x \mapsto x^2$ is convex, so $X(t)=Y^2(t)=(B^2(t)-t)^2$ is a submartingale. (Integrability is OK).

[Remark: A detailed calculation using conditional expectations or Itô's formula gives that $X(t)=(B^2(t)-\alpha t)^2$ is a submartingale if and only if $\alpha \in (-\infty, 0] \cup [1,3]$]

5) Pick $\alpha = -n$, so

$M(t) = B_1^2(t) + \dots + B_n^2(t) - n = (B_1^2(t) - 1) + (B_2^2(t) - 1) + \dots + (B_n^2(t) - 1)$ is

a martingale, $M(0) = 0$, $M(\tau) = |X(\tau)|^2 - n\tau = 1 - n\tau$.

Applying the optional stopping theorem (OST) gives

$$0 = EM(0) = EM(\tau) = 1 - nE\tau \Rightarrow \boxed{E\tau = \frac{1}{n}}.$$

Checking the conditions of the OST:

a.) $P(\tau > K) \le P(|B_1(1)| < 2, |B_1(2) - B_1(1)| < 2, \dots, |B_1(K) - B_1(K-1)| < 2) \underset{\text{increments}}{\overset{\text{independent}}{=}} [P(|\xi| < 2)]^K \to 0$ exponentially fast $\Rightarrow E\tau < \infty$.

$\xi \sim N(0,1)$

b.) If we define $Y(t) := |X(t)|^2$, then $Y(t \wedge \tau)$ is bounded, so its increments are bounded

$-n(t \wedge \tau)$ also has bounded increments $\Rightarrow$

$\Rightarrow$ ~~$M(t)$~~ $M(t \wedge \tau) = Y(t \wedge \tau) - n(t \wedge \tau)$ has bounded increments

$\Rightarrow$ the OST applies. $\square$