Stochastic differential equations
TU Budapest, spring semester 2019
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Working time: 90 minutes

In this exercise sheet, "Brownian motion" is used as a synonym of "Wiener process", and $B(t)$ denotes a standard 1-dimensional Brownian motion.

1. (6 points) Find the probability that there are real numbers $s, t \in \mathbb{R}$ with $0 \le s < t \le 1$ for which

   a.) $\frac{B(t)-B(s)}{t-s} > 100$

   b.) $\frac{|B(t)-B(s)|}{t-s} < \frac{1}{100}$

   c.) $|B(t) - B(s)| > (t-s)^{\frac{2}{3}}$

   Solution: Note that the question is not about a particular probability for given s and t, but the probability of *existence* of such s, t.

   a.) This asks the probability that $B(t)$ is not Lipschitz continuous with Lipschitz constant 100. This probability is 1, since we know (from the lecture) that $B(t)$ is almost surely nowhere Lipschitz continuous.

   b.) This asks the probability that there is an interval where the increment is small (compared to the length of the interval). This probability is 1, since we know (from the lecture) that $B(t)$ is almost surely nowhere monotone, so there are $s \neq t$ with $B(t) = B(s)$ (even in any small interval).

   c.) This asks the probability that $B(t)$ is not $\frac{2}{3}$-Hölder continuous with Hölder constant 1. This probability is 1, since we know (form HW 1.10 or 1.11) that $B(t)$ is almost surely nowhere $\frac{2}{3}$-Hölder continuous.

2. (4 points) Find values of $u, \alpha, \beta \in \mathbb{R}$ for which the process

   $$X(t) = B(u) - (\alpha + \beta t) B\left(\frac{1}{1-t}\right) \quad , \quad t \in [0,1]$$

   is a standard Brownian motion.

   Solution: $X(0) = B(u) - \alpha B(1)$. This has to be *identically* zero, so we must have $u = \alpha = 1$, so $X(t) = B(1) - (1-\beta t) B\left(\frac{1}{1-t}\right)$. At $t = 1$ the formula makes no sense, but the limit as $t \nearrow 1$ should exist. Then $\frac{1}{1-t} \to \infty$, so $B\left(\frac{1}{1-t}\right)$ is divergent: it has to be multiplied with something that goes to 0 if we want the product to be convergent. So $\beta = 1$ and

   $$X(t) = B(1) - (1-t) B\left(\frac{1}{1-t}\right)$$

   is the only possibility.

   To check that this is really a Brownian motion, we can refer to HW 1.6: part (a) says that $Y(t) := tB\left(\frac{1}{t}\right)$ is a standard Brownian motion, so part (c) says that $Z(t) = Y(1) - Y(1-t)$ is also a standard Brownian motion. For this,

   $$Z(t) = Y(1) - Y(1-t) = 1B\left(\frac{1}{1}\right) - (1-t) B\left(\frac{1}{1-t}\right) = X(t).$$

Alternatively, the fact that $X(t)$ is a standard Brownian motion can be checked by noting that it's a Gaussian process with mean 0 and calculating the covariances: Let's assume $0 < s < t$, then $1 < \frac{1}{1-s} < \frac{1}{1-t}$, so

$$\begin{aligned}
Cov(X(s), X(t)) =& Cov\left(B(1) - (1-s)B\left(\frac{1}{1-s}\right), B(1) - (1-t)B\left(\frac{1}{1-t}\right)\right) \\
=& Cov(B(1), B(1)) - (1-t)Cov\left(B(1), B\left(\frac{1}{1-t}\right)\right) \\
& - (1-s)Cov\left(B\left(\frac{1}{1-s}\right), B(1)\right) \\
& + (1-s)(1-t)Cov\left(B\left(\frac{1}{1-s}\right), B\left(\frac{1}{1-t}\right)\right) \\
=& 1 - (1-t)1 - (1-s)1 + (1-s)(1-t)\frac{1}{1-s} = s.
\end{aligned}$$

3. (8 points) For some fixed $x > 0$ let $\tau_x = \inf\{t > 0 \,|\, B(t) = x\}$ be the first hitting time of the point x. Calculate the density of the random variable τ_x.

   (Hint: the distribution function can be calculated using the reflection principle.)

   Solution: Let $M(t) = \max\{B(s) \,|\, 0 \le s \le t\}$ be the maximum of the Brownian motion on $[0, t]$. Then, since $x > 0$, we have $\tau_x \le t$ if and only if $M(t) \ge x$. So for $t > 0$ the distribution function of τ_x is

$$F_{\tau_x}(t) = \mathbb{P}(\tau_x \le t) = \mathbb{P}(M(t) \ge x).$$

   Now we know from the reflection principle that $\mathbb{P}(M(t) \ge x) = 2\mathbb{P}(B(t) \ge x)$. (This came from the fact that $\mathbb{P}(B(t) > gex \,|\, M(t) \ge x) = \frac{1}{2}$ and $\{B(t) \ge x\} \subset \{M(t) \ge x\}$.) So if ξ is standard Gaussian and Φ is the standard Gaussian distribution function, then

$$F_{\tau_x}(t) = 2\mathbb{P}(B(t) \ge x) = 2\mathbb{P}(\sqrt{t}\xi \ge x) = 2\mathbb{P}\left(\xi \ge \frac{x}{\sqrt{t}}\right) = 2\left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right).$$

   If ϕ denotes the standard normal density function, then the density of τ_x is

$$f_{\tau_x}(t) = \begin{cases} 0 & \text{if } t \le 0 \\ F'_{\tau_x}(t) = \cdots = \frac{1}{\sqrt{2\pi}} \frac{x}{t^{3/2}} e^{-\frac{x^2}{2t}} & \text{if } t > 0 \end{cases}.$$

   (Note that x is a parameter and f_{τ_x} is a function of t.)

4. (4 points) Find a nonzero deterministic function $f : \mathbb{R} \to \mathbb{R}$ for which the process $X(t) = f(t)e^{2B(t)}$ is a martingale.

   Solution 1: We know from class, and also from HW 2.6 that $t \mapsto e^{\theta B(t) - \frac{\theta^2}{2}t}$ is a martingale for every $\theta \in \mathbb{R}$. Choosing $\theta = 2$ gives that $t \mapsto e^{2B(t)-2t} = e^{-2t}e^{2B(t)}$ is a martingale, so $f(t) := e^{-2t}$ will do.

   Solution 2: $X(t) = F(t, B(t))$ where $F(t, x) = f(t)e^{2x}$, so

$$\begin{aligned}
\frac{\partial F}{\partial t} &= f'(t)e^{2x} \\
\frac{\partial F}{\partial x} &= 2f(t)e^{2x} \\
\frac{\partial^2 F}{\partial x^2} &= 4f(t)e^{2x}.
\end{aligned}$$

The Itô formula gives

$$dX(t) = f'(t)e^{2B(t)}\,\mathrm{d}t + 2f(t)e^{2B(t)}\,\mathrm{d}B(t) + \frac{1}{2}4f(t)e^{2B(t)}\,\mathrm{d}t$$
$$= [f'(t) + 2f(t)]\,e^{2B(t)} + 2X(t)\,\mathrm{d}B(t).$$

This means that $X(t)$ is a martingale if and only if $f'(t) + 2f(t) = 0$, which means

$$f(t) = const\, e^{-2t}.$$

5. (8 points) Let $X(t) = B(t) - t$. For $x \in \mathbb{R}$ let $\tau_x = \inf\{t > 0 \mid X(t) = x\}$ be the first hitting time of the point x. Calculate $\mathbb{E}\tau_x$ for every $x \in \mathbb{R}$.

   (Hint: a possible solution is to apply the optional stopping theorem to $M(t) = X(t) + \alpha t$ where $\alpha \in \mathbb{R}$ is chosen appropriately. If you do this, think of the conditions of the optional stopping theorem.)

   Solution: Choose α so that $M(t) = X(t) + \alpha t = B(t) + (\alpha - 1)t$ is a martingale, so $\alpha = 1$ and $M(t) = X(t) + t$. Applying the optional stopping theorem naively would give $\mathbb{E}M(\tau) = \mathbb{E}M(0) = 0$, which means in our case that $\mathbb{E}X(\tau) + \mathbb{E}\tau = 0$. Since $X(\tau) = x$, this gives that

$$\mathbb{E}\tau = -x.$$

   This is clearly nonsense when $x > 0$, since $\tau_x \geq 0$. But of course, $X(t) = B(t) - t$ has a *strong drift to the left*, so there is no guarantee that a fixed positive x is ever reached. Indeed, we know from HW 2.11 that if $x > 0$ then $\mathbb{P}(\tau_x = \infty) > 0$, so $\mathbb{E}\tau_x = \infty$, and the optional stopping theorem can not be applied. Taking that into consideration, we now have

$$\mathbb{E}\tau_x = \begin{cases} -x & \text{if } x \leq 0 \\ \infty & \text{if } x > 0 \end{cases}.$$

   (Remark: Making the argument rigorous by checking the conditions of the optional stopping theorem when $x < 0$ is more difficult, with many small arguments – I'm just writing it for completeness. A possible way is to fix $x < 0$ and add an $N > 0$ where we also stop the process, so consider $\tau := \min\{\tau_x, \tau_N\}$. This τ is easily seen to have finite expectation, moreover the stopped process $X(t \wedge \tau)$ is bounded (it stays in $[x, N]$). since $X(t \wedge \tau)$ is bounded, its increments $X((t+h) \wedge \tau) - X(t \wedge \tau)$ are also bounded, and so are the increments $(t + h) - t$ of the deterministic function t (with $h > 0$ fixed). So the stopped martingale $M(t \wedge \tau) = X(t \wedge \tau) - (t \wedge \tau)$ also has bounded increments, and the optional stopping theorem can be applied:

$$\mathbb{E}X(\tau) + \mathbb{E}\tau = \mathbb{E}M(\tau) = \mathbb{E}M(0) = 0.$$

   To calculate $\mathbb{E}X(\tau)$:

$$X(\tau) = \begin{cases} x & \text{if } \tau_x < \tau_N \\ N & \text{if } \tau_N < \tau_x \end{cases},$$

   so

$$\mathbb{E}X(\tau) = x\mathbb{P}(\tau_x < \tau_N) + N\mathbb{P}(\tau_N < \tau_x).$$

   We know from HW 2.11 that $\mathbb{P}(\tau_x < \tau_N) \to 1$ and $\mathbb{P}(\tau_N < \tau_x) \to 0$ exponentially fast as $N \to \infty$, so

$$\lim_{N\to\infty} \mathbb{E}X(\tau) = x,$$

   meaning

$$\lim_{N\to\infty} \mathbb{E}\tau = -x.$$

Finally, $\tau \nearrow \tau_x$ almost surely as $N \to \infty$, so the monotone convergence theorem guarantees that

$$\mathbb{E}\tau_x = \lim_{N\to\infty} \mathbb{E}\tau = -x.$$

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