Problem Set 3
Itô calculus

3.1 Let $s \mapsto v(s)$ be a smooth deterministic function with $\sup_{0 \le s \le T} |\, v'(s) \,| \le C$. Prove directly from the definition of the Itô integral that

$$\int_0^t v(s) dB(s) = v(t)B(t) - \int_0^t v'(s)B(s)ds.$$

Hint: Write

$$v(s_{i+1})B(s_{i+1}) - v(s_i)B(s_i) = v(s_i)(B(s_{i+1}) - B(s_i)) + B(s_{i+1})(v(s_{i+1})) - v(s_i)).$$

3.2 Prove directly form the definition of the Itô integral that

$$\int_0^t B(s)dB(s) = \frac{1}{2}B(t)^2 - \frac{t}{2},$$
$$\int_0^t B(s)^2 dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)ds.$$

3.3 Suppose $v, w \in \mathcal{V}_T$ and $C, D \in \mathbb{R}$ are such that

$$\int_0^T v(s)dB(s) + C = \int_0^T w(s)dB(s) + D.$$

Show that $C = D$ and $v = w$ (s, ω)-almost surely.

3.4 (a) For which values of $\alpha \in \mathbb{R}$ is the process

$$Y_\alpha(t) := \int_0^t (t-s)^{-\alpha} dB(s)$$

well defined as an Itô integral?.

(b) Compute the covariances $\mathbf{E}\,(Y_\alpha(s)Y_\alpha(t))$.

3.5 Use Itô's formula to write the following processes $t \mapsto X(t)$ in the standard form

$$X(t) = X(0) + \int_0^t u(s)ds + \int_0^t v(s)dB(s).$$

Identify the processes $s \mapsto u(s)$ and $s \mapsto v(s)$ under the integrals. Notation: $B(t)$ denotes standard 1-dimensional Brownian motion, $(B_1(t), \ldots, B_n(t))$ denotes standard n-dimensional Brownian motion (that is: n independent standard 1-dimensional Brownian motions).

(a) $X(t) = B(t)^2$

(b) $X(t) = 2 + t + e^{B(t)}$

(c) $X(t) = B_1(t)^2 + B_2(t)^2$

(d) $X(t) = (t, B(t))$

(e) $X(t) = (B_1(t) + B_2(t) + B_3(t), B_2(t)^2 - B_1(t)B_3(t))$

3.6 Use Itô's formula to prove that

$$\int_0^t B(s)^2 dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)ds.$$

3.7 Suppose $\theta(t) = (\theta_1(t), \ldots, \theta_n(t)) \in \mathbb{R}^n$ with $t \mapsto \theta_j(t)$, $j = 1, \ldots, n$, progressively measurable and a.s. bounded in any compact interval $[0, T]$. Define

$$Z(t) := \exp\left\{\int_0^t \theta(s)dB(s) - \frac{1}{2}\int_0^t |\, \theta(s) \,|^2 ds\right\},$$

where $t \mapsto B(t)$ is standard Brownian motion in $\mathbb{R}^n$ and $|\, \theta \,|^2 = \theta_1^2 + \cdots + \theta_n^2$.

(a) Use Itô's formula to prove that

$$dZ(t) = Z(t)\theta(t)dB(t).$$

(b) Deduce that $t \mapsto Z(t)$ is a martingale.

3.8 Let $t \mapsto B(t)$ be a standard 1-dimensional Brownian motion with $B(0) = 0$, and

$$\beta_k(t) := \mathbf{E}\left(B(t)^k\right).$$

Use Itô's formula to prove that

$$\beta_{k+2}(t) = \frac{1}{2}(k+2)(k+1)\int_0^t \beta_k(s)ds.$$

Compute explicitly $\beta_k(t)$ for $k = 0, 1, 2, \ldots, 6$.

3.9 Let $t \mapsto B(t)$ be a standard one-dimensional Brownian motion and $r, \alpha \in \mathbb{R}$ constants. Define

$$X(t) := \exp\{\alpha B(t) + rt\}.$$

Prove that

$$dX(t) = (r + \frac{\alpha^2}{2})X(t)dt + \alpha X(t)dB(t).$$

3.10 Let $t \mapsto B(t) \in \mathbb{R}^m$ be standard m-dimensional Brownian motion, $t \mapsto v(t) \in \mathbb{R}^{n\times m}$ progressively measurable and a.s. bounded. Define

$$X(t) = \int_0^t v(s)dB(s) \in \mathbb{R}^n.$$

Prove that

$$M(t) := |\, X(t) \,|^2 - \int_0^t \operatorname{tr}\{v(s)v(s)^T\}ds$$

is a martingale.

3.11 Use Itô's formula to prove that the following processes are $(\mathcal{F}_t^B)$-martingales.

(a) $X(t) = e^{t/2} \cos B(t)$

(b) $X(t) = e^{t/2} \sin B(t)$

(c) $X(t) = (B(t) + t) \exp\{-B(t) - t/2\}$

3.12 Let $t \mapsto u(t)$ be progressively measurable and almost surely bounded. Define

$$X(t) := \int_0^t u(s)ds + B(t),$$

$$M(t) := \exp\left\{ - \int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u(s)^2 ds \right\}$$

(Note that according to the statement of problem 7 the process $t \mapsto M(t)$ is a martingale.) Prove that the process

$$t \mapsto Y(t) := X(t)M(t)$$

is a $(\mathcal{F}_t^B)$-martingale.

3.13 In each of the cases below find a process $t \mapsto v(t)$ such that $v \in \mathcal{V}_T$ and the random variable X is written as

$$X = \mathbf{E}\,(X) + \int_0^T v(s)dB(s).$$

$$(a) \quad X = B(T), \qquad (b) \quad X = \int_0^T B(s)ds, \qquad (c) \quad X = B(T)^2,$$

$$(d) \quad B(T)^3, \qquad (e) \quad e^{B(T)}, \qquad (f) \quad \sin B(T).$$

3.14 Let $x \geq 0$ and define the process

$$X(t) := (x^{1/3} + \frac{1}{3}B(t))^3.$$

Show that

$$dX(t) = \frac{1}{3}\mathrm{sgn}(X(t)) \,|\, X(t) \,|^{1/3} \, dt + |\, X(t) \,|^{2/3} \, dB(t), \qquad X(0) = x.$$