Problem Set 5
Infinitesimal Generator, Dynkin's Formula

5.1 Write down the infinitesimal generator as elliptic differential operator for the following Itô diffusions:

(a) $dX(t) = \beta dt + \alpha X(t) dB(t)$.

(b) $dY(t) = \begin{pmatrix} dt \\ dX(t) \end{pmatrix}$, where $dX(t) = -\gamma X(t) dt + \alpha dB(t)$.

(c) $\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ X_2(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ e^{X_1(t)} \end{pmatrix} dB(t)$.

(d) $\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & X_1 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}$.

5.2 Find an Itô diffusion (i.e., write down the SDE for it) whose infinitesimal generator is the following:

(a) $Af(x) = f'(x) + f''(x)$, $f \in C_0^2(\mathbb{R})$.

(b) $Af(t,x) = \dfrac{\partial f}{\partial t} + cx\dfrac{\partial f}{\partial x} + \dfrac{1}{2}\alpha^2 x^2 \dfrac{\partial^2 f}{\partial x^2}$, $f \in C_0^2(\mathbb{R}^2)$.

5.3 Let $X(t)$ be a geometric Brownian motion, i.e. strong solution of the following SDE

$$dX(t) = \beta X(t) dt + \alpha X(t) dB(t), \qquad X_0 = x > 0,$$

where $\alpha > 0$, $\beta \in \mathbb{R}$ are fixed parameters.

(a) Find the generator A of the diffusion $t \mapsto X(t)$ and compute $Af(x)$ when $f : \mathbb{R}_+ \to \mathbb{R}$ is $f(x) = x^\gamma$, γ constant.

(b) Let $0 < r < R < \infty$, and $r \le x \le R$. using Dynkin's formula, compute

$$\mathbf{P}\big(\tau_r < \tau_R \,\big|\, X(0) = x\big),$$

where τ_r, and τ_R are the first hitting times of r, respectively, R.
Hint: Solve the boundary value problem $Af(x) = 0$ for $r < x < R$, with $f(r) = 1$, $f(R) = 0$.

(c) Assume $\beta < \alpha^2/2$. What is $\mathbf{P}\big(X(t) \text{ ever hits } R \,\big|\, X(0) = x\big)$?

(d) Assume $\beta > \alpha^2/2$. What is $\mathbf{P}\big(X(t) \text{ ever hits } r \bigm| X(0) = x\big)$?

5.4 (a) Find the generator of the δ-dimensional Bessel process, $BES(\delta)$

$$dY^{(\delta)}(t) = \frac{\delta - 1}{2Y^{(\delta)}(t)} dt + dB(t)$$

on $\mathbb{R}_+$.

(b) Let $0 < r < R < \infty$, and $r \le x \le R$. using Dynkin's formula, compute

$$\mathbf{P}\big(\tau_r < \tau_R \bigm| Y^{(\delta)}(0) = x\big),$$

where τ_r, and τ_R are the first hitting times of r, respectively, R.
Hint: Solve the boundary value problem $Af(x) = 0$ for $r < x < R$, with $f(r) = 1$, $f(R) = 0$. Note that the solutions are *qualitatively different* for $\delta \in [0, 2)$, $\delta = 2$, respectively, $\delta > 2$.

(c) Show that $BES(\delta)$ is transient if $\delta > 2$.

(d) Show that $BES(2)$ almost surely hits all points in $(0, \infty)$, but never hits 0.

(e) Show that for $\delta \in [0, 2)$ $BES(\delta)$ almost surely hits 0 (no matter where it starts).

5.5 Show that the solution $u(t, x)$ of the initial value problem

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2}\beta^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) + \alpha x \frac{\partial u}{\partial x}(t, x), \qquad t > 0,\ x \in \mathbb{R},$$
$$u(0, x) = f(x), \qquad (f \in C_K^2(\mathbb{R}) \text{ given})$$

can be expressed as follows:

$$\begin{aligned} u(t, x) &= \mathbf{E}\left(f\big(x \exp\{\beta B(t) + (\alpha - \beta^2/2)t\}\big)\right) \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f\big(x \exp\{\beta y + (\alpha - \beta^2/2)t\}\big) \exp(-y^2/(2t)) dy, \qquad t > 0\,. \end{aligned}$$

In this expression $t \mapsto B(t)$ is standard 1-dimensional Brownian motion with $B(0) = 0$.

5.6 Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function, for which $|\,\psi(x)\,| + |\,\nabla\psi(x)\,|^2 + |\,\Delta\psi(x)\,| \le C\,|\,x\,|^{2-\varepsilon}$. Prove that

$$M(t) := \exp\left\{\psi(B(t)) - \frac{1}{2}\int_0^t \big(|\,\nabla\psi(B(s))\,|^2 + \Delta\psi(B(s))\big) ds\right\}$$

is a martingale.