Problem Set 6
Strongly Continuous Contraction Semigroups and their Infinitesimal Generators

6.1 Let

$$\ell_\infty := \big\{ f : \mathbb{N} \to \mathbb{R} \, : \, \| f \| := \sup_{x \in \mathbb{N}} | f(x) | < \infty \big\},$$
$$c_0 := \big\{ f \in \ell_\infty \, : \, \lim_{x \to \infty} | f(x) | = 0, \; \| f \| := \sup_{x \in \mathbb{N}} | f(x) | \big\}.$$

Let $t \mapsto \eta_t \in \mathbb{N}$ be a time-homogeneous continuous time Markov chain on $\mathbb{N}$. Its transition operators are

$$P_t : \ell_\infty \to \ell_\infty, \qquad P_t f(x) := \mathbf{E}\big(f(\eta_t) \, \big| \, \eta_0 = x \big).$$

(a) Show that the one parameter family of operators $t \mapsto P_t$ form a semigroup of contractions on ℓ_∞.

(b) Give examples when $P_t : c_0 \to c_0$, and when $P_t : c_0 \not\to c_0$.

(c) Prove that if $P_t : c_0 \to c_0$ then by force the semigroup $t \mapsto P_t : c_0 \to c_0$ is strongly continuous.

(d) Give an example when $P_t : c_0 \not\to c_0$ and the semigroup $t \mapsto P_t : \ell_\infty \to \ell_\infty$ is strongly continuous.

(e) Give an example when the semigroup $t \mapsto P_t : \ell_\infty \to \ell_\infty$ is not strongly continuous.

6.2 Let $\mathcal{B}$ be a Banach space and $\mathcal{C} \subset \mathcal{B}$ a dense subspace. Recall that we call the densely defined operator $A : \mathcal{C} \to \mathcal{B}$ to be *dissipative* (or $-A$ to be *accretive*) if $\forall \varphi \in \mathcal{C}$ there exists a normalized tangent functional $\ell_\varphi \in \mathcal{B}^*$ to the vector φ, such that $\ell_\varphi(-A\varphi) \geq 0$. We showed in class that this implies that

$$\| (\lambda I - A)\varphi \| \geq \lambda \| \varphi \| , \qquad \text{for all } \varphi \in \mathcal{C}, \text{ and } \lambda > 0. \tag{1}$$

Conversely, if A is the infinitesimal generator of a strongly continuous contraction semigroup, then it is dissipative.

(a) Show that (1) implies that $A : \mathcal{C} \to \mathcal{B}$ is *closable.*

(b) Let

$$\begin{aligned}\mathcal{B} =& C_0[0,\infty) \\ :=& \{f : [0,\infty) \to \mathbb{R} \,:\, f \text{ continuous}, \ \lim_{x\to\infty} |\, f(x)\,| = 0, \ \text{with } \|\, f\,\| := \sup_{0\le x<\infty} |\, f(x)\,|\}.\end{aligned}$$

Consider $Af = \frac{1}{2}f''$ defined on

$$\widetilde{\mathcal{C}} := C_0[0,\infty) \cap C_0^2[0,\infty).$$

Show that A defined on $\widetilde{\mathcal{C}}$ does *not* satisfy (1).

(c) Show that, on the other hand, $Af = \frac{1}{2}f''$ defined on

$$\mathcal{C} := C_0[0,\infty) \cap C_0^2[0,\infty) \cap \{f'(0) = 0\}$$

does satisfy (1). The closure of this operator is the infinitesimal generator of Brownian motion on $[0,\infty)$ reflecting at 0.

6.3 Young's inequality for convolutions says that if $1 \le p, q, r \le \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, then

$$\|\, f * g\,\|_r \le \|\, f\,\|_p \,\|\, g\,\|_q \,.$$

Using this, show that $t \mapsto e^{\frac{1}{2}\Delta t}$ is a strongly continuous contraction semigroup on $\mathcal{L}^p$, $1 \le p < \infty$.

Hint: Use the explicit form of the heat-kernel:

$$e^{\frac{1}{2}\Delta t} f(x) = (2\pi t)^{d/2} \int_{\mathbb{R}^d} e^{|\,x-y\,|^2/2t} f(y)dy.$$

6.4 In this problem we consider the infinitesimal generator of Brownian motion in $\mathbb{R}^d$, that is: the Laplaciam Δ on the Banach space

$$\begin{aligned}\mathcal{B} =& C_0(\mathbb{R}^d) \\ :=& \{f : \mathbb{R}^d \to \mathbb{R} \,:\, f \text{ continuous}, \ \lim_{|\,x\,|\to\infty} |\, f(x)\,| = 0, \ \text{with } \|\, f\,\| := \sup_{x\in\mathbb{R}^d} |\, f(x)\,|\}.\end{aligned}$$

In $d = 1$ we have seen that the domain $\text{Dom}(\Delta) = C_0(\mathbb{R}) \cap C_0^2(\mathbb{R})$, i.e., vanishing value and vanishing 2nd derivative at infinity. We have also seen that on $\mathbb{R}^d$, $d \ge 2$, the Schwarz space $\mathcal{S}(\mathbb{R}^d)$ is a good core: the operator $-\Delta$ defined on $\mathcal{S}(\mathbb{R}^d)$ is dissipative, hence closable, and $\overline{\{\varphi - \Delta\varphi \,:\, \varphi \in \mathcal{S}(\mathbb{R}^d)\}} = C_0(\mathbb{R}^d)$, and thus Δ is indeed an infinitesimal generator, as we already knew. But what is $\text{Dom}(\Delta)$ obtained this way, in $\mathbb{R}^1$? I.e., what domain do we get when we close the operator from $\mathcal{S}(\mathbb{R}^1)$? It certainly contains $C_0(\mathbb{R}) \cap C_0^2(\mathbb{R})$, but isn't it larger?

6.5 (a) Let ψ be a bounded continuous function on $\mathbb{R}^d$, and $\lambda > 0$. Find a bounded solution u of the equation

$$\lambda u - \frac{1}{2}\Delta u = \psi \qquad \text{on } \mathbb{R}^d.$$

Prove that the solution is unique.

(b) Let $B(t)$ be d-dimensional Brownian motion ($d \geq 1$) and let F be a Borel set in $\mathbb{R}^d$. Let

$$T_F := |\{t \leq 1 : B(t) \in F\}|,$$

where $|\dots|$ denotes Lebesgue measure. Prove that $\mathbf{E}(T_F) = 0$ if and only if $|F| = 0$.

Hint: Consider the resolvent R_λ for $\lambda > 0$ and then let $\lambda \to 0$.)

6.6 In connection with the derivation of the Black-Scholes formula for the price of an option, the following partial differential equation appears for $u = u(t,x)$, $t \in [0,\infty)$, $x \in \mathbb{R}$:

$$\begin{aligned} &\frac{\partial u}{\partial t}(a,x) = -\rho u(t,x) + \alpha x \frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\beta^2 x^2 \frac{\partial^2 u}{\partial x^2}(t,x) && t > 0,\ x \in \mathbb{R} \\ &u(0,x) = (x-K)_+ && x \in \mathbb{R}, \end{aligned}$$

where $\rho > 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $K > 0$ are constants.

Use the Feynman-Kac formula to prove that the solution $u(t,x)$ of this initial value problem is given by

$$u(t,x) = \frac{e^{-\rho t}}{\sqrt{2\pi t}} \int_{\mathbb{R}} \big(xe^{(\alpha-\beta^2/2)t+\beta y} - K\big)_+ e^{-y^2/(2t)}\,dy, \qquad t > 0.$$

6.7 The elliptic Feynman-Kac formula, with Dirichlet boundary conditions.
Let $D \subset \mathbb{R}^d$ be a bounded domain with piecewise smooth boundary, $c, f : \mathbb{R}^d \to \mathbb{R}$ smooth functions and $c \geq 0$. Prove the following statement:
The unique solution of the elliptic boundary value problem

$$\begin{aligned} &\frac{1}{2}\Delta u - cu = f && \text{in } D \\ &u \equiv 0 && \text{on } \partial D, \end{aligned}$$

is given by

$$u(x) = \mathbf{E}\Big(\int_0^\tau f(B(t)) \exp\{-\int_0^t c(B(s))ds\} \mid B(0) = x\Big), \qquad x \in D,$$

where $B(t)$ is Brownian motion starting from $x \in D$ and τ is the first hitting time of ∂D.