Problem Set 7
Girsanov's Theorem and Some Applications

7.1 *[Change of conditional expectation]*
Let $\mathbf{Q}$ and $\mathbf{P}$ be two probability measures on $(\Omega, \mathcal{F})$, with $\mathbf{Q} \ll \mathbf{P}$, and Radon-Nikodym derivative $\frac{d\mathbf{Q}}{d\mathbf{P}}(\omega) = \varrho(\omega)$. Let $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra. Show that, for any $\mathcal{F}$-measurable random variable X, we have

$$\mathbf{E}_{\mathbf{Q}}\big(X \,\big|\, \mathcal{G} \big) = \frac{\mathbf{E}_{\mathbf{P}}\big(\varrho X \,\big|\, \mathcal{G} \big)}{\mathbf{E}_{\mathbf{P}}\big(\varrho \,\big|\, \mathcal{G} \big)} \tag{1}$$

7.2 *[A discrete version of Girsanov's formula]*
Let $\Omega_n := \{\mathsf{H}, \mathsf{T}\}^n$, $\mathbf{P}$ be the probability measure on Ω_n given by tossing a biased coin n times independently which gives probability $2/3$ to H, and $\mathbf{Q}$ the probability measure given by tossing a fair coin n times independently. Let $Z_n(\omega) := \frac{d\mathbf{Q}}{d\mathbf{P}}(\omega)$, and consider the martingale (with respect to the measure $\mathbf{P}$) $Z_m := \mathbf{E}_{\mathbf{P}}\big(Z_n \,\big|\, \mathcal{F}_m \big)$ for $m \leq n$.

(a) Give *explicitly* the distribution of Z_{m+1} given $Z_m, \ldots, Z_1$.

(b) Note that (1) of the previous exercise translates to $\mathbf{E}_{\mathbf{Q}}\big(X \,\big|\, \mathcal{F}_m \big) = (Z_m)^{-1}\mathbf{E}_{\mathbf{P}}\big(X Z_n \,\big|\, \mathcal{F}_m \big)$. Check this numerically for $n = 3$, $m = 2$, $X = \#\{\text{heads in } (\omega_1, \omega_2, \omega_3)\}$.

(c) Interpret this exercise as a discrete version of Girsanov's theorem.

7.3 *[Cameron-Martin theorem]*

(a) Let $f \in L^2[0,1]$ be a deterministic function and $F(t) := \int_0^t f(u)du$, $t \in [0,1]$. Show that, if $t \mapsto B(t)$ is standard 1d Brownian motion, then the laws of the processes $\{t \mapsto F(t)+B(t) : t \in [0,1]\}$ and $\{t \mapsto B(t) : t \in [0,1]\}$ are mutually absolutely continuous w.r.t. each other. Compute the Radon-Nikodym derivatives.

(b) If $F(t)$ is such that the above $f(t)$ does not exist, then the laws of the two processes are mutually singular.

7.4 Let $B(t) = (B_1(t), B_2(t))$, $t \leq T$, be a 2-dimensional standard Brownian motion on the probability space $(\Omega, \mathcal{F}_T, \mathbf{P})$. Find a probability measure $\mathbf{Q}$ on $\mathcal{F}_T$ that is mutually absolutely continuous w.r.t. $\mathbf{P}$, and under which the following process $t \mapsto Y(t)$ becomes a martingale:

(a)

$$dY(t) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} dt + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}, \qquad t \leq T.$$

(b)

$$dY(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}, \qquad t \leq T.$$

7.5 Let $b : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be bounded measurable function. Construct a *weak solution* $t \mapsto X(t)$ of the SDE

$$dX(t) = b(X(t))dt + dB(t), \qquad X_0 = x \in \mathbb{R}^n.$$

7.6 Let $B(t)$ be standard 1-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $Y(t) = t + B(t)$. For each $T > 0$, find $\mathbf{Q}_T \sim \mathbf{P}$ on $\mathbb{F}_T$ such that $\{t \mapsto Y(t)\}_{t \leq T}$ becomes a Brownian motion under $\mathbb{Q}_T$.

(a) Show that there exists a probability measure $\mathbf{Q}$ on $\mathcal{F}$ such that $\mathbf{Q}|_{\mathcal{F}_T} = \mathbb{Q}_T$ for all $T > 0$.

(b) Show that $\mathbf{P}\big(\lim_{t\to\infty} Y(t) = \infty\big) = 1$, while $\mathbf{Q}\big(\lim_{t\to\infty} Y(t) = \infty\big) = 0$. Why does not this contradict Girsanov's theorem?

7.7 Let $b : \mathbb{R} \to \mathbb{R}$ be Lipschitz, and $t \mapsto X(t)$ be the unique strong solution of the 1-dimensional SDE

$$dX(t) = b(X(t))dt + dB(t), \qquad X(0) = x \in \mathbb{R}.$$

(a) Use Girsanov's theorem to prove that for any $M < \infty$, $x \in \mathbb{R}$, and $t > 0$, we have $\mathbf{P}\,(X(t) > M) > 0$.

(b) Choose $b(x) = -r$, where $r > 0$ is a constant. Prove that, for all x, we have $\lim_{t\to\infty} X(t) = -\infty$, a.s. Compare this fact with the result in part (a).

7.8 *[Feynman-Kac formual and killing rates]*
Let $B(t)$ denote standard Brownian motion in $\mathbb{R}^n$, and consider the Itô diffusion

$$dX(t)t = \nabla h(X(t))dt + dB(t), \qquad X_0 = x \in \mathbb{R}^n,$$

where $h \in C^2_{\text{comp}}(\mathbb{R}^n)$. We are going to relate this process to a Brownian motion killed at a certain rate $V(x)$.

(a) Let

$$V(x) := \frac{1}{2}\,|\,\nabla h(x)\,|^2 + \frac{1}{2}\Delta h(x).$$

Prove that, for any $f \in C_{\text{comp}}(\mathbb{R}^n)$, we have

$$\mathbf{E}_x\big(f(X(t))\big) = \mathbf{E}_x\big(e^{-\int_0^t V(B(s))ds} e^{h(B(t))-h(x)} f(B(t))\big). \tag{2}$$

Hint: Use Girsanov's theorem to express the left hand side of (2) as an expectation with respect to $B(t)$, then use Itô's formula.

(b) Assume $V \geq 0$, and use Feynman-Kac with local killing rate $V(x)$. Let $Y(t)$ be the Brownian motion $B(t)$ *killed with local rate* $V(x)$. Reinterpret (2) as

$$P_t^X f(x) = e^{-h(x)} P_t^Y (e^h f)(x),$$

where $P_t^X f(x) := \mathbf{E}_x\big(f(X(t))\big)$ is the semigroup of conditional expectations for the diffusion $X(t)$, and similarly for Y.