Stochastic Processes, Problem Set 1: Solutions.

1.1

$$\frac{d}{dx}\left\{\left(\frac{1}{x}-\frac{1}{x^3}\right)\varphi(x)\right\} = \cdots = \left(-1+\frac{3}{x^4}\right)\varphi(x)$$

$$\vee$$

$$\frac{d}{dx}\left\{1-\Phi(x)\right\} = -\varphi(x)$$

$$\vee$$

$$\frac{d}{dx}\left\{\frac{1}{x}\varphi(x)\right\} = \cdots = \left(-1-\frac{1}{x^2}\right)\varphi(x)$$

Since

$$\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{x^2}\right)\varphi(x) = \lim_{x\to\infty}\left(1-\Phi(x)\right) = \lim_{x\to\infty}\frac{1}{x}\varphi(x) = 0$$

the claimed inequalities follow $\square$

1.2

a $$\mathbb{E}\left(B^{(n)}(t)\right) = \sum_{j=1}^{\lfloor nt\rfloor}\mathbb{E}\left(X_j^{(n)}\right) = 0$$

$$\mathrm{Cov}\left(B^{(n)}(t), B^{(n)}(s)\right) = \mathbb{E}\left(B^{(n)}(t)\, B^{(n)}(s)\right) =$$

$$= \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} \mathbb{E}\left(X_j^{(n)} X_i^{(n)}\right) = \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} \frac{1}{n} \delta_{ij}$$

$$= \frac{1}{n} \cdot \min\left(\lfloor nt \rfloor, \lfloor ns \rfloor\right) \to \min(s,t)$$

as $n \to \infty$.

(b) The collection of random variables $\{B^{(n)}(t) : t \in [0,1]\}$ is jointly Gaussian with expectations and covariances given in (a)

(c) $\delta_n = \max_{1 \le j \le n} |X_j^{(n)}|$

subadditivity of probabilities

$$\mathbb{P}\left(\max_{1 \le j \le n} |X_j^{(n)}| \ge \varepsilon\right) \le$$

$X_j^{(n)} \sim \frac{1}{\sqrt{n}} \xi$ $\leftarrow \mathcal{N}(0,1)$

$$\sum_{j=1}^{n} \mathbb{P}\left(|X_j^{(n)}| \ge \varepsilon\right) =$$

problem 1.1

$$= n\,\mathbb{P}\left(\frac{1}{\sqrt{n}} \xi \ge \varepsilon\right) = n\left(1 - \Phi(\sqrt{n}\,\varepsilon)\right) \le \ldots$$

$$n\left(1-\Phi(\sqrt{n}\,\varepsilon)\right) \leq \frac{\sqrt{n}}{\sqrt{2\pi}\,\varepsilon} e^{-\varepsilon^2 n/2} \longrightarrow 0$$

by problem 1.1

as $n \to \infty$

$\square$

1.3

a $$\mathbb{E}\left(Z^{(n)}(t)\right) = \sum_{j=1}^{\lfloor nt \rfloor} \mathbb{E}\left(Y_j^{(n)} - \frac{1}{n}\right) = 0$$

$$\mathrm{Cov}\left(Z^{(n)}(t), Z^{(n)}(s)\right) = \sum_{j=1}^{\lfloor nt \rfloor} \sum_{i=1}^{\lfloor ns \rfloor} \mathrm{Cov}\left(Y_j^{(n)}, Y_i^{(n)}\right) =$$

$$= \sum_{j=1}^{\lfloor nt \rfloor} \sum_{i=1}^{\lfloor ns \rfloor} \frac{1}{n} \delta_{i,j} = \frac{1}{n} \min\left(\lfloor ns \rfloor, \lfloor nt \rfloor\right) \xrightarrow[\text{as } n \to \infty]{} \min(s,t)$$

b Let $0 = t_0 < t_1 < t_2 < \dots < t_n = 1$

Then

$\left\{ Z^{(n)}(t_j) - Z^{(n)}(t_{j-1}) \;:\; j = 1, \dots, n \right\}$ are independent

and the distribution of $Z^{(n)}(t) - Z^{(n)}(s)$ is

$\mathrm{POI}\left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)$ centred (expectation subtracted)

(c) $\delta_n = \max_{1 \le j \le n} Y_j^{(n)} - \frac{1}{n}$

for any $\varepsilon < 1$: let $n > \frac{1}{1-\varepsilon}$

Then

$$\mathbb{P}(\delta_n > \varepsilon) = \mathbb{P}\left(\max_{1 \le j \le n} Y_j^{(n)} \geq 1\right) =$$

$$1 - \mathbb{P}\left(Y_j^{(n)} = 0,\ j = 1, \dots, n\right) = 1 - \left(e^{-\frac{1}{n}}\right)^n$$

$$= 1 - e^{-1}$$

$\square$

(1.4) $B^{(n)}(t)$ is "almost continuous"

while

$Z^{(n)}(t)$ has jumps of order 1

$B^{(n)}(t) \Longrightarrow B(t)$ while $Z^{(n)}(t) \Rightarrow \text{Poisson}(t)$

in some sense

in some sense

1.5 $C = (c_{ij})_{i,j=1}^{n}$ $\quad c_{ij} = \mathrm{Cov}(Y_i, Y_j) = \mathbb{E}(Y_i Y_j)$

C is $\boxed{\text{positive definite}}$

$\exists!\ A = (a_{ij})_{i,j=1}^{n}$ $\boxed{\text{positive definite}}$ matrix, real

such that $C = A^2$. Denote $B := A^{-1}$

if $\underline{Y} = (Y_1, \dots Y_n)^T$ is jointly Gaussian

let $\underline{\underline{X}} = B\,\underline{Y}$ $\quad \underline{X} = (X_1, \dots, X_n)^T$

As linear combinations of jointly Gaussians, $(X_i)_{i=1}^{n}$ <u>are</u> jointly Gaussian, too.

$$\mathbb{E}(X_i X_j) = \sum_{k,l} B_{ik} B_{jl}\ \mathbb{E}(Y_k Y_l) =$$

$$(BCB)_{ij} = \delta_{ij}$$

So the Gaussian random variables $(X_i)_{i=1}^{n}$ are uncorrelated and thus independent

1.6 $X(t)$, $Y(t)$, $Z(t)$ are Gaussian processes since they are <u>linear</u> expressions of $B(t)$.

$$\mathbb{E}(X(t)) = \mathbb{E}(Y(t)) = \mathbb{E}(Z(t)) = 0 \checkmark$$

$$\mathbb{E}(X(t)X(s)) = a^{-1}\,\mathbb{E}(B(at)B(as)) =$$

$$a^{-1}\min(as, at) = \min(s, t) \checkmark$$

$$\mathbb{E}(Y(t)Y(s)) = t \cdot s\; \mathbb{E}(B(1/t)\,B(1/s))$$

$$= t\,s \cdot \min\left(\frac{1}{t}, \frac{1}{s}\right) = \min(t, s)$$

$$\mathbb{E}(Z(t)Z(s)) = \mathbb{E}\Big(\big(B(T) - B(T-t)\big)\big(B(T) - B(T-s)\big)\Big)$$

$$= \mathbb{E}(B(T)B(T)) - \mathbb{E}(B(T)B(T-s)) -$$

$$- \mathbb{E}(B(T-t)B(T)) + \mathbb{E}(B(T-t)B(T-s)) =$$

$$T - (T-s) - (T-t) + \min(T-t, T-s) =$$

$$= \min(t, s) \qquad \square$$

Continuity:

$t \mapsto X(t)$ is continuous on $t \in [0,\infty)$

$t \mapsto Y(t)$ — " — on $t \in (0,\infty)$

$t \mapsto Z(t)$ — " — on $t \in [0,T]$

straightforward

Continuity of $t \mapsto Y(t)$ at $t=0$ needs proof.

Since $t \mapsto Y(t)$ is continuous on $(0,\infty)$

$$\left\{\lim_{t \downarrow 0} Y(t) = 0\right\} =$$

$$\bigcap_{m>0} \bigcup_{n>0} \bigcap_{q \in (0,\frac{1}{n}) \cap \mathbb{Q}} \left\{ |Y(q)| < \frac{1}{n} \right\}$$

Since $(Y(t) : t > 0) \sim (B(t) : t > 0)$

it follows that

$$\mathbb{P}\left(\lim_{t \downarrow 0} Y(t) = 0\right) = \mathbb{P}\left(\lim_{t \downarrow 0} B(t) = 0\right) = \frac{1}{0}$$

17 straightforward: $Z(t)$ is Gaussian

$$E\big(Z(t)Z(s)\big) = \sum_{ij} a_i a_j \underbrace{E\big(B_i(t)B_j(s)\big)}_{=\delta_{ij}\min(t,s)} =$$

$$= \sum_{i=1}^{n} a_i^2 \min(t,s)$$

So, $t \mapsto Z(t)$ is BM with

$$\sigma^2 = \sum_{i=1}^{n} a_i^2 \qquad \square$$

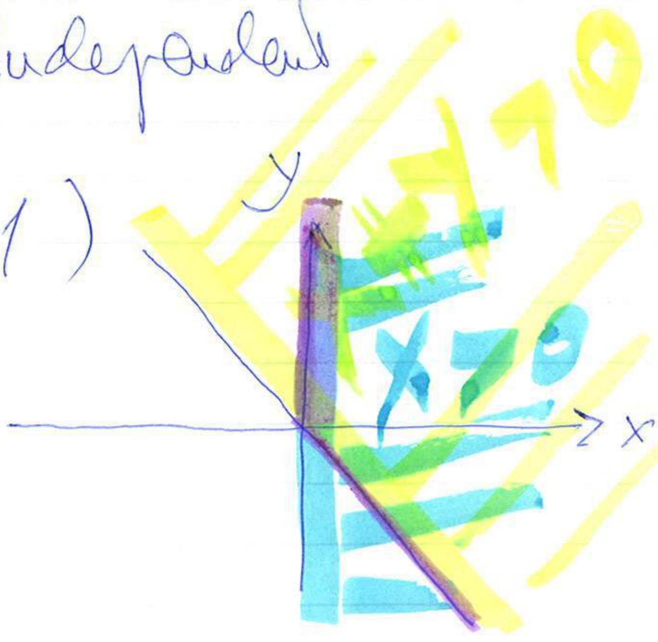
18 $$P\Big(\big(B(2)-B(1)\big)+B(1)\geq 0 \;\Big|\; B(1)\geq 0\Big) =$$

$$\mathbb{P}\big(X+Y \geq 0 \mid X \geq 0\big)$$

where X, Y are independent

$$\sim \mathcal{N}(0,1)$$

$$= \frac{3}{4} \text{ see picture}$$



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$$\mathbb{P}\left(\inf_{0\le t\le 1} B(t)=0\right)=$$

$$\mathbb{P}\left(\bigcap_n\left\{\inf_{0\le t\le 1} B(t)>-\frac{1}{n}\right\}\right)=$$

$$\lim_{n\to\infty}\mathbb{P}\left(\inf_{0\le t\le 1} B(t)\ge-\frac{1}{n}\right)=$$

$$\lim_{n\to\infty}\mathbb{P}\left(M(1)\le\frac{1}{n}\right)=$$

$$\lim_{n\to\infty}\left(2\Phi\left(\frac{1}{n}\right)-1\right)=0.$$

similarly : $\mathbb{P}\left(\sup_{0\le t\le 1} B(t)=0\right)=0$

$$\{B(t) \text{ doesn't change sign in } [0,1]\}=$$

$$\left\{\inf_{0\le t\le 1} B(t)=0\right\}\cup\left\{\sup_{0\le t\le 1} B(t)=0\right\}$$

Thus:

$\mathbb{P}(B(t)$ doesn't change sign in $[0,1])=0.$

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let $\varepsilon > 0$. By scaling

$\mathbb{P}(B(t)$ doesn't change sign in $[0,\varepsilon]) = 0$

$\mathbb{P}(B(t)$ doesn't change sign on $[0,\varepsilon]$ for some $\varepsilon > 0)$

$= \mathbb{P}\left(\bigcup_n \{B(t) \text{ doesn't change sign in } [0,\tfrac{1}{n}]\}\right) =$

$= \lim_{n\to\infty} \mathbb{P}(B(t)$ doesn't change sign in $[0,\tfrac{1}{n}]) = 0.$

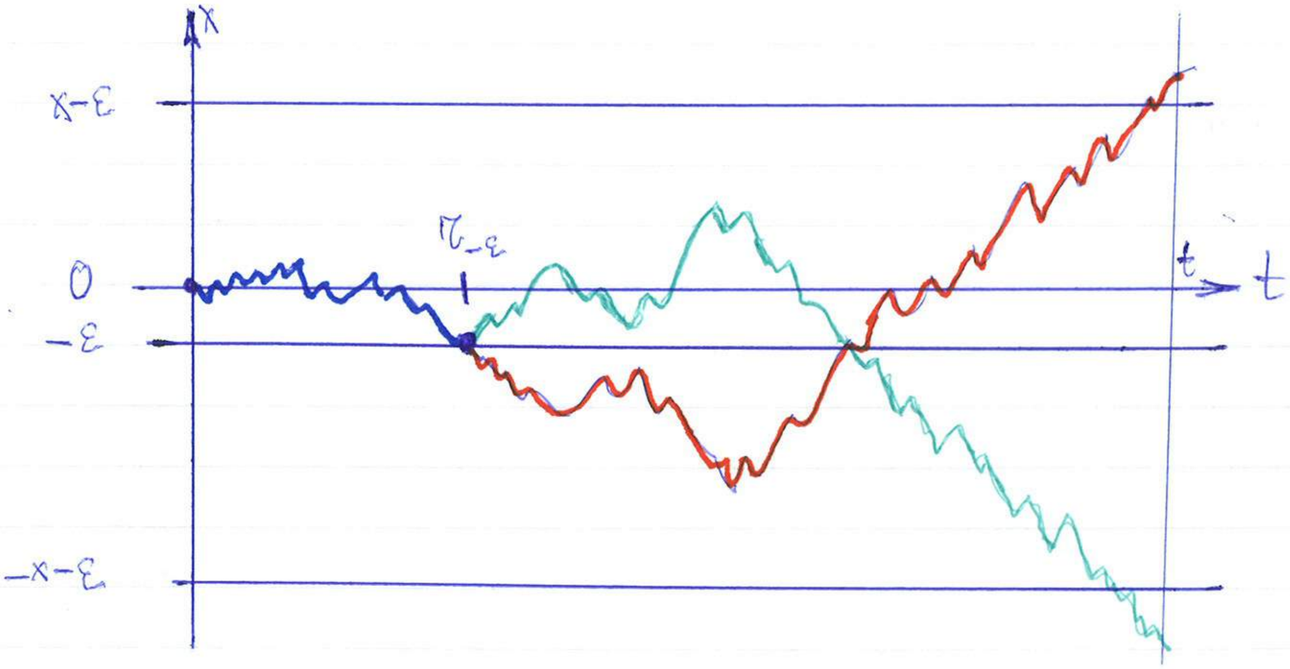
so: $\mathbb{P}\left(\inf\{t > 0 : B(t) = 0\} = 0\right) = 1.$

1.10 **a** $\mathbb{P}\left(\min_{0\le s\le t} B(s) \ge -\varepsilon\right) \overset{\mathrm{I}}{=} 2\Phi(\varepsilon/\sqrt{t}) - 1$

see distribution of the maximum

$\mathbb{P}\left(\{B(t) \ge x-\varepsilon\} \cap \{\min_{0\le s\le t} B(s) \ge -\varepsilon\}\right) =$

$$\mathbb{P}\big(B(t) \geq x-\varepsilon\big) - \underbrace{\mathbb{P}\Big(\{B(t) \geq x-\varepsilon\} \cap \{\min_{0\leq s\leq t} B(s) < -\varepsilon\}\Big)}_{\substack{= \text{ reflection principle, see picture} \\ \mathbb{P}(B(t) < -x-\varepsilon)}}$$



Thus:

$$\mathbb{P}\Big(\{B(t) \geq x-\varepsilon\} \cap \{\min_{0\leq s\leq t} B(s) \geq -\varepsilon\}\Big) \overset{\text{II}}{=}$$

$$\Phi\left(\frac{-x+\varepsilon}{\sqrt{t}}\right) - \Phi\left(\frac{-x-\varepsilon}{\sqrt{t}}\right)$$

I & II yield $(*)$ of the problem.

b letting $\varepsilon \to 0$ in $(*)$ we get:

$$\mathbb{P}\left(B(t) \geq x \,\middle|\, \min_{0 \leq s \leq t} B(s) \geq 0\right) = \frac{\varphi(x/\sqrt{t})}{\varphi(0)} = e^{-x^2/2t}$$

differentiating:

$$-\frac{\partial}{\partial x} \mathbb{P}\left(B(t) \geq x \,\middle|\, \min_{0 \leq s \leq t} B(s) \geq 0\right) = \frac{x}{t} e^{-x^2/2t} \, \mathbb{1}(x \geq 0). \quad \square$$

1.11 $K\psi_n(t) =$

$$\sqrt{2}\int_0^t s \sin\left(\frac{\pi(2n-1)}{2} s\right) ds +$$

$$\sqrt{2}\int_t^1 t \sin\left(\frac{\pi(2n-1)}{2} s\right) ds$$

$$\int_0^t s \sin\left(\frac{\pi(2n-1)}{2} s\right) ds =$$

integrate by parts

$$-\frac{2}{\pi(2n-1)}\left[s \cdot \cos\left(\frac{\pi(2n-1)}{2} s\right)\right]_0^t +$$

$$\frac{2}{\pi(2n-1)}\int_0^t \cos\left(\frac{\pi(2n-1)}{2} s\right) ds =$$

$$-\frac{2}{\pi(2n-1)} t \cdot \cos\left(\frac{\pi(2n-1)}{2} t\right) + \frac{4}{\pi^2(2n-1)^2} \sin\left(\frac{\pi(2n-1)}{2} t\right)$$

$$\int_t^1 \sin\left(\frac{\pi(2n-1)}{2} s\right) ds = \frac{2}{\pi(2n-1)} \cdot \cos\left(\frac{\pi(2n-1)}{2} t\right)$$

putting these together we get: $K\psi_n = \lambda_n \psi_n$ $\square$

1.12

$$\boxed{\lambda < 1}$$

$$E\left(e^{\lambda(\xi^2-1)/2}\right) =$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{\lambda(x^2-1)/2} dx =$$

$$e^{-\lambda/2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1-\lambda}{2}x^2} dx =$$

$$e^{-\lambda/2} \frac{1}{\sqrt{1-\lambda}}$$

$$\psi(\lambda) = \log\left(\qquad \right) = -\frac{1}{2}\left(\log(1-\lambda) + \lambda\right)$$

$$\psi'(\lambda) = \frac{1}{2}\left(\frac{1}{1-\lambda} \quad 1\right) \ldots$$

$$= \frac{1}{2} \frac{\lambda}{1-\lambda}$$

$$\psi''(\lambda) = \frac{1}{2} \frac{1}{(1-\lambda)^2} > 0$$

ψ is convex

graph: in lecture notes

1.13

$$\frac{\partial \phi}{\partial t}(t,x) = -\frac{1}{2} t^{-3/2}\left(\varphi\left(\frac{x}{\sqrt{t}}\right) + \frac{x}{\sqrt{t}}\varphi'\left(\frac{x}{\sqrt{t}}\right)\right)$$

$$= -\frac{1}{2} t^{-3/2}\left(1 - \frac{x^2}{t}\right)\varphi\left(\frac{x}{\sqrt{t}}\right)$$

$$\frac{\partial^2 \phi}{\partial x^2}(t,x) = t^{-3/2}\varphi''\left(\frac{x}{\sqrt{t}}\right)$$

$$= t^{-3/2}\left(\frac{x^2}{t} - 1\right)\varphi\left(\frac{x}{\sqrt{t}}\right)$$

$\square$