Stochastic Processes / M6006
Problem set 3 – Solutions

3.1 $0 = s_0 < s_1 < \dots < s_n = t$

$$\max_{0 \leq i < n} |s_{i+1} - s_i| \longrightarrow 0 \quad \text{as } n \to \infty$$

$$v(t)B(t) - v(0)B(0) =$$

$$\sum_{i=0}^{n-1} \big(v(s_{i+1})B(s_{i+1}) - v(s_i)B(s_i) \big) =$$

$$\sum_{i=0}^{n-1} v(s_i)\big(B(s_{i+1}) - B(s_i)\big) +$$

$$\sum_{i=0}^{n-1} B(s_{i+1}) \underbrace{\big(v(s_{i+1}) - v(s_i)\big)}_{= v'(s^*_{i+1})(s_{i+1}-s_i)\,*} = *$$

turn page

with some $s_i \leq s^*_{i+1} \leq s_{i+1}$

$$v(t)B(t) - v(0)B(0) =$$

$$\left.\sum_{i=0}^{n-1} v(s_i)\left(B(s_{i+1}) - B(s_i)\right)\right\} \xrightarrow{\text{Itô}} \int_0^t v(s)\,dB(s)$$

$$\left.+ \sum_{i=0}^{\infty} B(s_{i+1})\, v'(s_{i+1}^*)\left(s_{i+1} - s_i\right)\right\} \longrightarrow \int_0^t B(s)v'(s)\,ds$$

Riemann

$$= \int_0^t v(s)\,dB(s) + \int_0^t B(s)v'(s)\,ds$$

3.2 $B(t)^2 - B(0)^2 =$

$$\sum_{i=0}^{n-1}\left(B(s_{i+1})^2 - B(s_i)^2\right) =$$

$$2\sum_{i=0}^{n-1} B(s_i)\left(B(s_{i+1}) - B(s_i)\right) \xrightarrow[L^2]{\text{Itô}} 2\int B(s)\,dB(s) \quad +$$

$$+ \sum_{i=1}^{n-1}\left(B(s_{i+1}) - B(s_i)\right)^2 \xrightarrow[\text{quadr. var.}]{L^2} t$$

$$B(t)^3 - B(0)^3 =$$

$$\sum_{i=0}^{n-1} \left(B(s_{i+1})^3 - B(s_i)^3 \right) =$$

$$3 \sum_{i=0}^{n-1} B(s_i)^2 \left(B(s_{i+1}) - B(s_i) \right) \xrightarrow{L^2} 3 \int_0^t B(s)^2 \, dB(s) \quad \text{Itô}$$

$$+ 3 \sum_{i=0}^{n-1} B(s_i) \left(B(s_{i+1}) - B(s_i) \right)^2 \xrightarrow{L^2} 3 \int_0^t B(s) \, ds$$

$$+ \sum_{i=0}^{n-1} \left(B(s_{i+1}) - B(s_i) \right)^3 \xrightarrow{L^2} 0$$

$$\mathbb{E} \left\{ \sum_{i=0}^{n-1} B(s_i) \left(\left(B(s_{i+1}) - B(s_i) \right)^2 - (s_{i+1} - s_i) \right) \right\}^2 =$$

$$\sum_{i=0}^{n-1} \mathbb{E} \left(B(s_i)^2 \left(\left(B(s_{i+1}) - B(s_i) \right)^2 - (s_{i+1} - s_i) \right)^2 \right) +$$

$$2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \mathbb{E} \left(B(s_i) B(s_j) \left(\left(B(s_{i+1}) - B(s_i) \right)^2 - (s_{i+1} - s_i) \right) \left(\left(B(s_{j+1}) - B(s_j) \right)^2 - (s_{j+1} - s_j) \right) \right) \to 0$$

$$= C \sum_{i=0}^{n-1} s_i (s_{i+1} - s_i)^2 \longrightarrow 0.$$

the other term done similarly.

This ↓ is problem 2.4 now, Go further down for 3.3

2.4.

(a) $$E\left(B(t) + t \mid \mathcal{F}_s^B\right) = B(s) + t$$

$$\neq B(s) + s$$

not martingale

(b) $$E\left(B(t)^2 \mid \mathcal{F}_s^B\right) = B(s)^2 + (t-s)$$

$$\neq B(s)^2$$

not martingale

(c) $$E\left(t^2 B(t) - 2\int_0^t B(\cdot) \mid \mathcal{F}_s^B\right) =$$

$$t^2 B(s) - 2\int_0^s r B(r)\,dr - 2\int_s^t r B(s)\,dr$$

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$$= s^2 B(s) - 2\int_0^s r B(r)\,dr$$

this one is a martingale

(d) $$\mathbb{E}\left(B_1(t)B_2(t) \,\middle|\, \mathcal{F}_s^B\right) = B_1(s)B_2(s)$$

it is a martingale

3.3 $$\xi := C - D + \int_0^T \big(v(s) - w(s)\big)\,dB(s)$$

$$\xi = 0$$

But $$\mathbb{E}(\xi) = C - D$$

$$\mathrm{Var}(\xi) = \int_0^T \mathbb{E}\left(|v(s) - w(s)|^2\right) ds$$

both must vanish. Hence the claim.

3.4

$$\mathbb{E}\left(Y_\alpha(t)^2\right) = \int_0^t (t-s)^{-2\alpha}\, ds < \infty$$

If $\alpha < \frac{1}{2}$

$0 < s < t$

$$\mathbb{E}\left(Y_\alpha(s)Y_\alpha(t)\right) =$$

$$\mathbb{E}\left(\int_0^s (s-u)^{-\alpha}\, dB(u) \int_0^t (t-r)^{-\alpha}\, dB(r)\right) =$$

$$\int_0^s (s-u)^{-\alpha}(t-u)^{-\alpha}\, du =$$

$$s^{1-2\alpha} \int_0^1 (1-u)^{-\alpha}\left(\frac{t}{s}-u\right)^{-\alpha} du.$$

Stochastic Processes – Problem Set 3 continued

Solutions

3.5

(a) $dX(t) = 2B(t)\,dB(t) + dt$

$X(0) = 0$

$u(s) \equiv 1$

$v(s) = 2B(s)$

(b) $dX(t) = dt + e^{B(t)}dB(t) + \frac{1}{2}e^{B(t)}dt$

$X(0) = 2, \; u(s) = 1 + \frac{1}{2}e^{B(s)}, \; v(s) = e^{B(s)}$

(c) $dX(t) = 2B_1(t)\,dB_1(t) + 2B_2(t)\,dB_2(t)$

$+ dt + dt$

$X(0) = 0, \; u(s) \equiv 2 \; ; \; v(s) = \begin{pmatrix} 2B_1(s) \\ 2B_2(s) \end{pmatrix}$

(d)

$$dX(t) = \begin{pmatrix} dt \\ dB(t) \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;;\; u(s) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;;\; v(s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(e)

$$dX(t) = \begin{pmatrix} dB_1(t) + dB_2(t) + dB_3(t) \\ 2B_2(t)dB_2(t) + dt - B_1(t)dB_3(t) - B_3(t)dB_1(t) \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;;\; u(s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$v(s) = \begin{pmatrix} 1 & 1 & 1 \\ -B_3(s) & 2B_2(s) & -B_1(s) \end{pmatrix}$$

3.6 $d\left(B(t)^3\right) = 3B(t)^2 dB(t) + 3B(t)\,dt$

37 (a) $Y(t) = \int_0^t \theta(s) \cdot dB(s) - \frac{1}{2} \int_0^t |\theta(s)|^2 ds$

$$Z(t) = \exp(Y(t))$$

$$dZ(t) = \exp(Y(t))\, dY(t) + \frac{1}{2} \exp(Y(t)) \underbrace{(dY(t))^2}_{= |\theta(t)|^2 dt}$$

$$= \dots = \exp(Y(t))\, \theta(t) \cdot dB(t)$$

$$= Z(t)\, \theta(t) \cdot dB(t)$$

(b) thus

$$Z(t) = \int_0^t Z(s)\, \theta(s) \cdot dB(s)$$

is an Itô integral $\Longrightarrow$ martingale.

3.8 $d\left(B^{k+2}(t)\right) =$

$$(k+2)\,B(t)^{k+1}\,dB(t) + \frac{(k+2)(k+1)}{2}\,B(t)^{k}\,dt$$

$$B(t)^{k+2} = (k+2)\int_0^t B(s)^{k+1}\,dB(s) + \frac{(k+2)(k+1)}{2}\int_0^t B(s)^{k}\,ds$$

Hence

$$\mathbb{E}\left(B(t)^{k+2}\right) = \overbrace{(k+2)\,\mathbb{E}\left(\int_0^t B(s)^{k+1}\,dB(s)\right)}^{=0} + \frac{(k+2)(k+1)}{2}\int_0^t \mathbb{E}\left(B(s)^{k}\right)ds.$$

Hence $\mathbb{E}\left(B(t)^{k+2}\right) = \dfrac{(k+2)(k+1)}{2}\displaystyle\int_0^t \mathbb{E}\left(B(s)^{k}\right)ds$

Hence $\mathbb{E}\left(B(t)^{2k+1}\right) = 0 \quad \forall k$

$$\mathbb{E}\left(B(t)^2\right) = t$$

$$\mathbb{E}\left(B(t)^4\right) = \frac{4\cdot 3}{2}\int_0^t s\,ds = 3t^2$$

$$\mathbb{E}\left(B(t)^6\right) = \frac{6\cdot 5}{2}\int_0^t 3\cdot s^2\,ds = 15t^3$$

$$\mathbb{E}\left(B(t)^{2k}\right) = \frac{(2k)!}{2^k k!}\cdot t^k$$

Induction

$$\mathbb{E}\left(B(t)^{2(k+1)}\right) = \frac{2k!}{2^k k!}\,\frac{(2k+2)(2k+1)}{2}\int_0^t s^k\,dk$$

$$= \frac{2(k+1)!}{2^{k+1}\,(k+1)!}\,t^{k+1}$$

✓

3.9

$$dX(t) = e^{(\alpha B(t)+rt)}\cdot(\alpha\,dB(t)+r\,dt)$$

$$+\frac{1}{2}e^{(\alpha B(t)+rt)}\cdot(\alpha\,dB(t)+r\,dt)^2 =$$

$$= \dots = \left(r+\frac{\alpha^2}{2}\right)X(t)dt + \alpha X(t)dB(t).$$

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3.10 $$dM(t) = 2X(t)\cdot dX(t) + |dX(t)|^2 - tr(v(t)\,v(t)^T)\,dt$$

$$= 2X(t)\cdot v(t)\, dB(t)$$

$$M(t) = M(0) + 2\int_0^t X(s)\cdot v(s)\, dB(s)$$

is an Itô integral

Hence: a martingale

3.11 (a) $$dX(t) = \frac{1}{2} e^{t/2} \cos B(t)\, dt - e^{t/2} \sin B(t)\, dB(t) - \frac{1}{2} e^{t/2} \cos B(t) \underbrace{(dB(t))^2}_{dt}$$

$$= -e^{t/2} \sin B(t)\, dB(t)$$ ✓

(b) very similar

(c) $$dX(t) = \left(1 - \frac{1}{2}(B(t)+t)\right) e^{-(B(t)+t/2)} dt$$

$$+ \left(1 - (B(t)+t)\right) e^{-(B(t)+t/2)} dB(t)$$

$$- \frac{1}{2}\left(1 + 1 - B(t) - t\right) e^{-(B(t)+t/2)} (dB(t))^2$$

$$= \left(1 - B(t) - t\right) e^{-(B(t)+t/2)} dB(t)$$

3.12 $$dX(t) = u(t)dt + dB(t)$$

$$dM(t) = -M(t)\, u(t) \cdot dB(t)$$

$$dY(t) = X(t)\,dM(t) + M(t)\,dX(t)$$

$$+ dX(t)\,dM(t) =$$

$$= -X(t)M(t)u(t)dB(t) +$$

$$M(t)u(t)dt + M(t)dB(t) +$$

$$\underbrace{(u(t)dt + dB(t))(-M(t)u(t)dB(t))}_{= -M(t)u(t)dt}$$

$$= \left(M(t) - M(t)X(t)u(t)\right)dB(t)$$

3.13

(a) $X = \int_0^T dB(s)$

(b) $X = \int_0^T B(s)\,ds = \int_0^T (T-s)\,dB(s)$

(c) $X = B(T)^2 = T + 2\int_0^T B(s)\,dB(s)$

(d) $X = B(T)^3 = 3\int_0^T (B(s)^2 + T - s)\,dB(s)$

(e) $e^{B(T) - \frac{T}{2}} = A + \int_0^T e^{B(s) - \frac{s}{2}}\,dB(s)$

$$e^{B(T)} = e^{\frac{T}{2}} + \int_0^T e^{B(s) + \frac{T-s}{2}}\,dB(s)$$

(f) $d\, e^{\frac{t}{2}} \sin B(t) = e^{\frac{t}{2}} \cos B(t)\,dB(t)$

$$\sin B(T) = \int_0^T e^{(t-T)/2} \cos B(t)\,dB(t)$$

3.14

$$dX(t) = \left(X^{1/3} + \frac{1}{3}B(t)\right)^2 dB(t) + \frac{1}{3}\left(X^{1/3} + \frac{1}{3}B(t)\right) dt$$

$$= |X(t)|^{2/3} dB(t) + \frac{1}{3}\,\mathrm{sgn}(X(t))\,|X(t)|^{1/3} dt$$

$\square$