# Problem Set 4 : SDE-s
## Solutions

**4.1** Straightforward applications of Itô's formula.

**4.2** $\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = \begin{pmatrix} \cos B(t) \\ \sin B(t) \end{pmatrix}$

by applying Itô's formula one gets

$$d\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} dt + \begin{pmatrix} -V(t) \\ U(t) \end{pmatrix} dB(t).$$

4.3

a) $dB(t) = e^{t} dX(t) + e^{t} X(t)\,dt$

$\overset{\text{Itô}}{=} d\left(e^{t} X(t)\right)$

$X(t) = e^{-t} X(0) + e^{-t} B(t)$

b) Let $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} := \begin{pmatrix} r \cdot t + \alpha \int_0^t X(s)\,dB(s) \\ B(t) \end{pmatrix}$

and apply Itô to $g(t; x, y) = x\, e^{-\alpha y + \frac{\alpha^2}{2} t}$

$Z(t) = g(t; X(t), B(t))$

$dZ(t) = \frac{\alpha^2}{2} X(t)\, e^{-\alpha B(t) + \frac{\alpha^2}{2} t}$

$+ e^{-\alpha B(t) + \frac{\alpha^2}{2} t}\, dX(t)$

$$- \alpha X(t) e^{-\alpha B(t) + \frac{\alpha^2}{2} t} dB(t)$$

$$+ \frac{\alpha^2}{2} X(t) e^{-\alpha B(t) + \frac{\alpha^2}{2} t} dt$$

$$- 2 \cdot \frac{\alpha^2}{2} X(t) e^{-\alpha B(t) + \frac{\alpha^2}{2} t} dt =$$

$$e^{-\alpha B(t) + \frac{\alpha^2}{2} t} \left( dX(t) - \alpha X(t) dB(t) \right) =$$

$$r e^{-\alpha B(t) + \frac{\alpha^2}{2} t} dt$$

We arrived at

$$d \left( X(t) e^{-\alpha B(t) + \frac{\alpha^2}{2} t} \right) = r e^{-\alpha B(t) + \frac{\alpha^2}{2} t} dt$$

$$\boxed{X(t) = X(0) e^{\alpha B(t) - \frac{\alpha^2}{2} t} + r \int_0^t e^{\alpha (B(t) - B(s)) - \frac{\alpha^2}{2}(t-s)} ds}$$

c)
$$d\left(e^{-\mathcal{J}t}X(t)\right) = e^{-\mathcal{J}t}A\,dB(t)$$

$$X(t) = e^{\mathcal{J}t}X(0) + \int_0^t e^{\mathcal{J}(t-s)}A\,dB(t)$$

Note that: $e^{\mathcal{J}t} = \begin{pmatrix} \cos t & +\sin t \\ -\sin t & \cos t \end{pmatrix}$

$$\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} \cos t\, X_1(0) + \sin t\, X_2(0) \\ -\sin t\, X_1(0) + \cos t\, X_2(0) \end{pmatrix}$$

$$+ \int_0^t \underbrace{\begin{pmatrix} \cos(t-s), & +\sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}}_{\begin{pmatrix} \alpha\cos(t-s), & +\beta\sin(t-s) \\ -\alpha\sin(t-s), & \beta\cos(t-s) \end{pmatrix}} \begin{pmatrix} dB_1(s) \\ dB_2(s) \end{pmatrix}$$

4.4

a $X(t) = X(0)e^{-\gamma t} + a\int_0^t e^{-\gamma(t-s)} dB(s)$

b $E(X(t)) = x_0 e^{-\gamma t}$

let $0 \leq s \leq t < \infty$ :

$Cov(X(t), X(s)) =$

$a^2 E\left(\int_0^t e^{-\gamma(t-u)} dB(u) \cdot \int_0^t e^{-\gamma(s-v)} dB(v)\right) =$

$a^2 \int_0^s e^{-\gamma(t+s-2u)} du =$

$\frac{a^2}{2\gamma}\left(e^{-\gamma|t-s|} - e^{-\gamma(t+s)}\right)$

(c) $$\mathbb{E}\left( Y^{(n)}_{k+1} - Y^{(n)}_{k} \,\middle|\, Y^{(n)}_{k} = y \right) =$$

$$- \frac{2}{n} \left( y - \frac{n}{2} \right)$$

$$\mathrm{Var}\left( Y^{(n)}_{k+1} - Y^{n}_{k} \,\middle|\, Y^{(n)}_{k} = y \right) =$$

$$1 - \frac{4}{n^2} \left( y - \frac{n}{2} \right)^2$$

May think about it as:

$$Y^{(n)}_{k+1} - Y^{(n)}_{k} = - \frac{2}{n} \left( Y^{(n)}_{k} - \frac{n}{2} \right) + \xi^{(n)}_{k+1} +$$

$$+ \text{small error} \quad \cancel{\xi^{(n)}_{k+1}}$$

random variable independent of the past

$$\mathbb{E}\left(\xi^{(n)}_{k+1}\right) = 0$$

$$\mathrm{Var}\left(\xi^{(n)}_{k+1}\right) = 1$$

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Now, let $X^{(n)}(t) := \frac{1}{\sqrt{n}}\left(Y^{(n)}_{\lfloor nt \rfloor} - \frac{n}{2}\right)$

$$dt \simeq \frac{1}{N}$$

to get

$$dX^{(n)}(t) \approx -2X^{(n)}(t)\,dt + dB(t) \ldots$$

4.5 $t \mapsto X(t) \quad t \in [0,1]$

Gaussian, continuous sample paths

$\mathbb{E}(X(t)) = 0$

$\mathbb{E}(X(t)X(s)) = (t \wedge s)(1-(t \vee s))$

a) $t \mapsto X(t)$ clearly Gaussian, continuous

$\mathbb{E}(X(t)) = \mathbb{E}(B(t)) - t\mathbb{E}(B(1)) = 0$

$0 \leq s \leq t \leq 1$

$$E(X(t)X(s)) = E(B(t)B(s))$$
$$- t\,E(B(1)B(s)) - s\,E(B(t)B(1))$$
$$+ st\,E(B(1)B(1)) = \dots = s(1-t)$$

b) similarly:

$$E(Y(t)) = (1-t)\,E\left(B\left(\frac{t}{1-t}\right)\right) = 0$$

$0 \leq s \leq t \leq 1$

$$E(Y(s)Y(t)) =$$
$$(1-t)(1-s)\,E\left(B\left(\frac{t}{1-t}\right)B\left(\frac{s}{1-s}\right)\right) =$$
$$\dots (1-t) \cdot s.$$

c) similarly:

$$E(Z(t)) = 0$$

$$E(Z(s)Z(t)) = \dots$$

$$= \int_0^s \frac{(1-t)}{(1-u)} \cdot \frac{(1-s)}{(1-u)}\, du =$$

$$= (1-t)(1-s) \int_0^s \frac{du}{(1-u)^2} = \dots = s(1-t)$$

d) differentiate $\rightarrow$ $\boxed{\dfrac{Z(t)}{1-t} = \int_0^t \dfrac{dB(s)}{1-s}}$

Continuity at $t \to 0$:

argument on blackboard ...

Continuity of $t \mapsto Y(t)$ (and $t \mapsto Z(t)$) at $t = 1$:

$t \mapsto M(t) := (1-t)^{-1} Y(t)$ is a continuous martingale for $t \in [0,1)$

for any $0 \leq t \leq u < 1$

$$\mathbb{P}\Big(\sup_{t \leq s \leq u} |Y(s)| \geq \lambda\Big) \leq$$

$$\mathbb{P}\Big(\sup_{t \leq s \leq u} |M(s)| \geq (1-u)^{-1}\lambda\Big) \overset{\text{by Doob's maximal ineq}}{\leq}$$

$$\frac{(1-u)^2}{\lambda^2} \mathbb{E}\big(M(u)^2\big) = \frac{u(1-u)}{\lambda^2} \qquad (*)$$

Let $t_n := 1 - 2^{-n}$ $\quad n = 0, 1, 2, \dots$

$$A_n := \Big\{\sup_{t_{n-1} \leq s \leq t_n} |Y(s)| > 2^{-\frac{n}{4}}\Big\}$$

Then by $\circledast$

$$\mathbb{P}(A_n) \leq 2^{-\frac{n}{2}}$$ summable

Thus, by Borel-Cantelli, almost surely

$\exists\, n_0 < \infty$ (random, but a.s. finite!)

such that

$(\forall n > n_0)$ :

$$\sup_{1-2^{-(n+1)} \leq s \leq 1-2^{-n}} |Y(s)| \leq 2^{-\frac{n}{4}}$$

This implies: $\lim_{t \to 1} Y(t) = 0$, a.s.

$\square$.

4.6



$t \mapsto \theta(t) = (\theta_1(t), \dots, \theta_n(t))$

deterministic, piece-wise continuous, bdd.

$$M_\theta(t) := \exp\left\{ \int_0^t \theta(s) \cdot dY(s) - \frac{1}{2} \int_0^t |\theta(s)|^2 ds \right\}$$

is a martingale.

Since $s \mapsto \theta(s)$ is deterministic

we get:

$$\mathbb{E}\left( e^{\int_0^t \theta(s) dY(s)} \right) = e^{\frac{1}{2} \int_0^t |\theta(s)|^2 ds}$$

let e.g. $\theta_i(s) = \chi_{[a_i, b_i]}(s) \cdot \lambda_i$

$$\mathbb{E}\left(e^{\sum_{i=1}^{n} \lambda_i \left(Y_i(b_i) - Y_i(a_i)\right)}\right) = e^{\frac{1}{2}\sum_{i=1}^{n} \lambda_i^2 (b_i - a_i)}$$

these are exactly the exponential moments of Gaussians ...

4.7 ...