Problem Set 7 - Solutions

7.1 Assume first that $\mathcal{G}$ is discrete with partition elements $(G_k)_{k=1,2,\dots}$

Then

$$E_Q(X|G_k) = \frac{\int_{G_k} X(\omega)\, dQ(\omega)}{\int_{G_k} dQ(\omega)} =$$

$$= \frac{\int_{G_k} X(\omega)\, g(\omega)\, dP(\omega)}{\int_{G_k} g(\omega)\, dP(\omega)} \cdot \frac{P(G_k)}{P(G_k)}$$

$$= \frac{E_P(X \cdot g | G_k)}{E_P(g | G_k)} \quad \dots$$

General case (conditional expectation à la Kolmogorov)

let $Z \in \mathcal{L}^\infty(\Omega, \mathcal{G}, \mathbb{P})$ (bdd, $\mathcal{G}$-measurable)

$$\mathbb{E}_{\mathbb{Q}}\left(Z \cdot \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G})\right) = \mathbb{E}_{\mathbb{Q}}(ZX)$$

by def. of conditional expectation

$$= \mathbb{E}_P(\varrho Z \cdot X)$$

$$\mathbb{E}_{\mathbb{Q}}\left(Z \cdot \frac{\mathbb{E}_P(\varrho X|\mathcal{G})}{\mathbb{E}_P(\varrho|\mathcal{G})}\right) = \mathbb{E}_P\left(\varrho Z \frac{\mathbb{E}_P(\varrho X|\mathcal{G})}{\mathbb{E}_P(\varrho|\mathcal{G})}\right)$$

$$= \mathbb{E}_P\left(\varrho \frac{\mathbb{E}_P(\varrho Z X|\mathcal{G})}{\mathbb{E}_P(\varrho|\mathcal{G})}\right) =$$

$$= \mathbb{E}_P\left(\mathbb{E}_P(\varrho|\mathcal{G}) \cdot \frac{\mathbb{E}_P(\varrho Z X|\mathcal{G})}{\mathbb{E}_P(\varrho|\mathcal{G})}\right) =$$

$$= \mathbb{E}_P\left(\mathbb{E}_P(\varrho Z X|\mathcal{G})\right) = \mathbb{E}_P(\varrho Z X) \checkmark$$

7.2 $\Omega_n = \{0,1\}^n \qquad T=0,\ H=1$

$$\mathbb{P}_n(\omega_1\omega_2\cdots\omega_n) = \left(\frac{2}{3}\right)^{\sum_1^n \omega_k} \left(\frac{1}{3}\right)^{\sum_1^n (1-\omega_k)}$$

$$\mathbb{Q}_n(\omega_1\omega_2\cdots\omega_n) = \left(\frac{1}{2}\right)^n$$

$$\frac{d\mathbb{Q}_n}{d\mathbb{P}_n}(\underline{\omega}) = Z_n(\underline{\omega}) = \left(\frac{3}{2}\right)^n \cdot (2)^{-\sum_1^n \omega_k}$$

$$Z_{m+1} = Z_m \cdot \frac{3}{2} \cdot 2^{-\omega_m}$$

a) given $(Z_1, Z_2, \ldots Z_m)$, $Z_{m+1} = \begin{cases} Z_m \cdot \frac{3}{2} & \text{with prob. } \frac{1}{3} \\ Z_m \frac{3}{4} & \text{with prob. } \frac{2}{3} \end{cases}$

$\boxed{\text{under } \mathbb{P}}$

b) $$\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{F}_2) = \omega_1 + \omega_2 + \frac{1}{2}$$

$$\cancel{\mathbb{E}_p(X \cdot Z \mid \mathcal{F}_2) = \mathbb{E}_p\left((\omega_1+\omega_2+\omega_3)\left(\frac{3}{2}\right)^3\right)}$$

$$\mathbb{E}_p\left(X \cdot \frac{Z_3}{Z_2} \,\middle|\, \mathcal{F}_2\right) = \mathbb{E}_p\left((\omega_1+\omega_2+\omega_3) \cdot \frac{3}{2} \cdot 2^{-\omega_3} \,\middle|\, \mathcal{F}_2\right)$$

(4)

$$= \frac{3}{2}(\omega_1+\omega_2)\underbrace{E_p\left(2^{-\omega_3}\middle|\mathcal{F}_2\right)}_{=2/3} +$$

$$\frac{3}{2}\underbrace{E_p\left(\omega_3 2^{-\omega_3}\middle|\mathcal{F}_2\right)}_{=\frac{1}{3}}$$

$$= \omega_1+\omega_2 + \frac{1}{2} \checkmark$$

(c) ... interpretation...

(7.3) (a) Follows from Girsanov

$$X(t) := F(t) + B(t)$$

$$E\left(\Phi\left(X(u): 0\le u\le T\right)\right) =$$

$$E\left(\Phi\left(B(u): 0\le u\le T\right)e^{\int_0^T f(u)dB(u) - \frac{1}{2}\int_0^T (f(u))^2 du}\right)$$

(b) if $\int |f(u)|^2 = \infty$ then the R.N derivative doesn't exists

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7.4 a

$$dY(t) = \underbrace{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}_{\mu} dt + \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{\nu} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}$$

$$r = \nu^{-1}\mu = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$M(t) = \exp\left\{-3B_1(t) + B_2(t) - 5t\right\}$$

$$dQ_t = M(t)\, dP_t$$

b

$$dY(t) = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mu} dt + \underbrace{\begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}}_{\nu} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}$$

$$r = \nu^{-1}\mu = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$M(t) = \exp\left\{3B_1(t) - B_2(t) - 5t\right\}$$

$$dQ_t = M(t)\, dP_t$$

F.S $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$

$t \mapsto B(t) \in \mathbb{R}^n \qquad (\mathcal{F}_t)_t$ - B.M.

$$M(t) := \exp\left\{ \int_0^t b(B(u)) \cdot dB(u) - \frac{1}{2} \int_0^t |b(B(u))|^2 du \right\}$$

$$d\mathbb{Q}_t := M(t)\, d\mathbb{P}_t$$

<u>then</u>: under the measure $\mathbb{Q}_T$

$$t \mapsto X(t) := B(t)$$

is <u>weak</u> solution of

$$dX(t) = b(X(t))dt + d\hat{B}(t)$$

where $d\hat{B}(t) = dB(t) - b(B(t))dt$

is BM under $\mathbb{Q}$.

7.6 $Y(t) = B(t) + t$

$M(t) = \exp\left\{-B(t) - \frac{t}{2}\right\}$

$d\mathbb{Q}_t = M(t)\, d\mathbb{P}_t$

$t \mapsto Y(t)$ - is $\boxed{\text{drifted}}$ BM under $\mathbb{P}$

- is standard BM under $\mathbb{Q}$

$\boxed{\mathbb{Q}_\infty \text{ and } \mathbb{P}_\infty \text{ are mutually singular}}$

7.7 $\mathbb{P}(X(t) > M) > 0$

$\Updownarrow$

$\mathbb{Q}(X(t) > M) > 0$

$\parallel$

$\mathbb{P}(B(t) > M)$

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7.8 $h \in C_0^2(\mathbb{R}^n)$

$$dX(t) = \nabla h(X(t))\,dt + dB(t), \quad X(0) = x$$

$t \mapsto X(t)$ strong sln.

$$M(t) = \exp\left\{-\int_0^t \nabla h(X(u)) \cdot dB(u) - \frac{1}{2}\int_0^t |\nabla h(X(u))|^2 du\right\}$$

$$= \exp\left\{-\int_0^t \nabla h(X(u)) \cdot dX(u) + \frac{1}{2}\int_0^t |\nabla h(X(u))|^2 du\right.$$

a

$$\mathbb{E}_x(f(X(t))) = \mathbb{E}_x\left(M(t)\, M(t)^{-1} f(X(t))\right)$$

$$\overset{\text{Girsanov}}{=} \mathbb{E}_x\left(e^{\int_0^t \nabla h(\hat{B}(u)) \cdot d\hat{B}(u) - \frac{1}{2}\int_0^t |\nabla h(\hat{B}(u))|^2 du} f(\hat{B}(t))\right)$$

$$\overset{\text{Itô}}{=} e^{-h(x)}\, \mathbb{E}_x\underbrace{\left(e^{-\frac{1}{2}\int_0^t (|\nabla h(\hat{B}(u))|^2 + \Delta h(\hat{B}(u)))du}\, e^{h(\hat{B}(t))} f(\hat{B}(t))\right)}_{v(t,x) :=}$$

b) by Feynman-Kac formula:

$v(t,x) :=$

$$\mathbb{E}_x\left(e^{-\frac{1}{2}\int_0^t \left(|\nabla h(B(u))|^2 + \Delta h(B(u))\right)du}\, e^{h(B(t))} f(B(t))\right)$$

is the unique bdd. solution of the initial value problem

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{1}{2}\Delta v(t,x) - V(x)\, v(t,x) \\ v(0,x) = e^{h(x)} f(x) \end{cases}$$