Balint Toth:

Brownian Motion/3

Further properties of BM

Properties of the distribution (law):

① Reflection invariance:

have the same law

$$\left(t \mapsto -B(t)\right) \sim \left(t \mapsto B(t)\right)$$

② Scaling: let $a > 0$

$$\left(t \mapsto a^{-1/2} B(at)\right) \sim \left(t \mapsto B(t)\right)$$

③ Time inversion:

$$\left(t \mapsto t\,B(1/t)\right) \sim \left(t \mapsto B(t)\right)$$

Proof: Since on the l.h.s. we have linear transformations of the process, these processes are also Gaussian.

HW Check expectations and covariances!

Continuity of paths is preserved

3 at $t=0$ needs (easy) argument

due to continuity: $\{\lim_{t\downarrow 0} B(t)=0\} = \bigcap_{m\geq 1}\bigcup_{n\geq 1}\{|B(t)|<\frac{1}{m} \text{ for all } t\in(0,1/n]\cap\mathbb{Q}\}$ same prob $=1$

$\{\lim_{t\downarrow 0} X(t)=0\} = \bigcap_{m\geq 1}\bigcup_{n\geq 1}\{|X(t)|<\frac{1}{m} \text{ for all } t\in(0,1/n]\cap\mathbb{Q}\}$

Sample path properties:

No intervals of monotonicity (easy):

Thm Almost surely, for any $0\leq a<b<\infty$

soft $t\mapsto B(t)$ is not monotone in $[a,b]$.

Proof:

$$\mathbb{P}(t\mapsto B(t) \text{ monotone in } [0,1])=0$$

Since

$$\mathbb{P}(t\mapsto B(t) \text{ monotone increasing on } [0,1]) \leq$$

$$\mathbb{P}\left(B\left(\tfrac{i+1}{n}\right)-B\left(\tfrac{i}{n}\right)\geq 0, i=0,1,\dots,n-1\right) = 2^{-n}\to 0.$$

indep + stationary increments

$p, q \in \mathbb{Q}$, $0 \leq p < q$

$$\mathbb{P}\left(t \mapsto B(t) \text{ monotone on } [p,q]\right) = 0$$

by stati. increments + scaling

$$\mathbb{P}\left(\exists\, p,q \in \mathbb{Q} \quad 0 \leq p < q : t \mapsto B(t) \text{ mono on } [p,q]\right)$$
$$= 0$$

since $\mathbb{Q}$ countable / subadditivity of $\mathbb{P}$.

$\square$

<u>No differentiability at deterministic t:</u> (easy)

Thm (soft) Let $h : [0,1) \to \mathbb{R}_+$ continuous, monotone incr, $h(0) = 0$.
(e.g. $h(t) = t^{\alpha}$, $\alpha > 0$; $h(t) = |\log t|^{-1}$, ...)
Almost surely

$$\overline{\lim_{t \downarrow 0}} \frac{|B(t)|}{\sqrt{t}\, h(t)} = \infty.$$

<u>Proof</u>: $\mathbb{P}\left(\frac{|B(t)|}{\sqrt{t}\,h(t)} < c\right) =$

$$\mathbb{P}\left(|\xi| \leq c\,h(t)\right) \leq \frac{c}{\sqrt{2\pi}}\,h(t)$$

$\mathcal{N}(0,1)$

Choose $t_n \to 0$ so that

$$\sum_{n=1}^{\infty} h(t_n) < \infty$$

By Borel-Cantelli a.s. $\exists\ n_0 = n_0(\omega)$ so that for $n \geq n_0$, $\frac{B(t)}{\sqrt{t}\,h(t)} > c$

Hence: a.s. $\overline{\lim}\, \frac{B(t)}{\sqrt{t}\,h(t)} \geq c$ $\square$

<u>Remarks</u> (1) in particular: no differentiability at t.

(2) We'll see much stronger bounds.

Law of the iterated logarithm (LIL)

(subtle) at typical times t:

Thm For any $t \geq 0$, almost surely

$$-1 = \underline{\lim}_{h \to 0} \frac{B(t+h)-B(t)}{\sqrt{2h\log\log(1/h)}} \leq \overline{\lim}_{h\to 0} \frac{B(t+h)-B(t)}{\sqrt{2h\log\log(1/h)}} = 1$$

Modulus of continuity (subtle)

Thm Almost surely

$$\overline{\lim}_{h\to 0} \sup_{0\leq t\leq 1} \frac{|B(t+h)-B(t)|}{\sqrt{2h\log(1/h)}} = 1.$$

(at exceptional - random - times)

Nowhere differentiability of BM

Thm (Paley, Wiener, Zygmund, 1933)

The BM is almost surely nowhere differentiable

(No exceptional – random – points where it happens to be diff'ble)

Proof (following Dvoretzky - Erdős - Kakutani 1961)

$$\{t \mapsto B(t) \text{ diff'ble at (some) } t^* \in [0,1]\} \subseteq$$

$$\left\{ \overline{\lim_{h\to 0}} \frac{|B(t^*+h)-B(t^*)|}{|h|} < \infty \right\} =$$

$$\bigcup_{M\in\mathbb{N}} \left\{ \overline{\lim_{h\to 0}} \frac{|B(t^*+h)-B(t^*)|}{|h|} < M \right\} \subseteq$$

$$\bigcup_{M\in\mathbb{N}} \left\{ \exists \varepsilon > 0 : |h| \leq \varepsilon \Rightarrow \left|B(t^*+h)-B(t^*)\right| \leq 2M|h| \right\} \subseteq$$

$$\bigcup_{M\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \left\{ \exists j \in \{3,\dots,m\} : \max\left\{ \left|B\left(\tfrac{j-2}{m}\right)-B\left(\tfrac{j-3}{m}\right)\right|, \left|B\left(\tfrac{j-1}{m}\right)-B\left(\tfrac{j-2}{m}\right)\right|, \left|B\left(\tfrac{j}{m}\right)-B\left(\tfrac{j-1}{m}\right)\right| \right\} \leq \frac{4M}{m} \right\}$$

Call the event in the previous line $A_{M,m}$

We prove $\lim_{m\to\infty} \mathbb{P}(A_{M,m}) = 0$,

hence $\mathbb{P}\left(\bigcup_{M\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_{M,m}\right) = 0$.

$$\mathbb{P}\left(\bigcup_{j=3}^{m} \left\{\max_{k=1,2,3} \left|B\left(\frac{j-k+1}{m}\right) - B\left(\frac{j-k}{m}\right)\right| \leq \frac{4M}{m}\right\}\right) \leq$$

$$\sum_{j=3}^{m} \mathbb{P}\left(\max_{k=1,2,3} \left|B\left(\frac{j-k+1}{m}\right) - B\left(\frac{j-k}{m}\right)\right| \leq \frac{4M}{m}\right) =$$

$$(m-2)\,\mathbb{P}\left(|\xi| \leq \frac{4M}{\sqrt{m}}\right)^3 \leq$$

$$\left(\frac{8M}{\sqrt{2\pi}}\right)^3 m^{-1/2} \longrightarrow 0$$

use: independent and stationary increments + scaling of BM

$\square$