Bálint Tóth: Brownian Motion/4

Quadratic variation of BM

$0 < T < \infty$ fixed, $B(t)$, $t \in [0,T]$
standard 1-d BM

$0 = s_{n,0} < s_{n,1} < \dots < s_{n,n-1} < s_{n,n} = T$
$n \in \mathbb{N}$ sequence of subdivisions of $[0,T]$
(deterministic)

Notation:

$\Delta s_{n,j} := s_{nj} - s_{n,j-1}$ $\qquad j = 1, 2, \dots, n$

$\Delta B_{n,j} := B(s_{n,j}) - B(s_{n,j-1})$

$$\delta_n := \max_{1 \le j \le n} \Delta s_{nj}$$

assumed: $\lim_{n \to \infty} \delta_n = 0$

$$V_{n,\alpha} := \sum_{j=1}^{n} \left| \Delta B_{n,j} \right|^{\alpha}, \quad 0<\alpha<\infty$$

$\alpha = 1$ "total variation" (usual)
$\alpha = 2$ "quadratic variation"

Theorem:

(i) If $\boxed{\alpha<2}$, then $\forall M<\infty$

$$\mathbb{P}\left(V_{n,\alpha} < M\right) \to 0 \text{ as } n\to\infty,$$

or $V_{n,\alpha} \xrightarrow{\mathbb{P}} \infty$.

(ii) If $\boxed{\alpha=2}$ then $\forall \varepsilon>0$

$$\mathbb{P}\left(\left|V_{n,2} - T\right| > \varepsilon\right) \to 0, \text{ as } n\to\infty,$$

or $V_{n,2} \xrightarrow{P} T$ (we have seen this)

(iii) If $\boxed{\alpha>2}$, then $\forall \varepsilon>0$

$$\mathbb{P}\left(V_{n,\alpha} > \varepsilon\right) \to 0, \text{ as } n\to\infty,$$

or $V_{n,\alpha} \xrightarrow{P} 0$

Lemma: Let $\beta \in [0, \infty)$

① $T \delta_n^{\beta-1} \leq \sum_{j=1}^{n} |\Delta s_{nj}|^\beta \leq T^\beta n^{1-\beta}$ for $\boxed{0 \leq \beta \leq 1}$

② $T^\beta n^{1-\beta} \leq \sum_{j=1}^{n} |\Delta s_{nj}|^\beta \leq T \delta_n^{\beta-1}$ for $\boxed{1 \leq \beta < \infty}$

③ $$n^{-1} \leq \frac{\sum_{j=1}^{n} |\Delta s_{nj}|^{2\beta}}{\left(\sum_{j=1}^{n} |\Delta s_{nj}|^\beta\right)^2} \leq \delta_n T^{-1}$$ $\boxed{\text{for } 0 \leq \beta \leq 1}$

Proof (of Lemma)

① lower

$$\sum_{j=1}^{n} |\Delta s_{nj}|^\beta = \sum_{j=1}^{n} |\Delta s_{n,j}| \cdot \underbrace{|\Delta s_{nj}|^{\beta-1}}_{\geq \delta_n^{\beta-1}}$$ since $\beta < 1$

$$\geq \delta_n^{\beta-1} \sum_{j=1}^{n} |\Delta s_{nj}| = \delta_n^{\beta-1} \cdot T$$

1 upper

$$\sum_{j=1}^{n} |\Delta s_{nj}|^{\beta} = n \sum_{j=1}^{n} \frac{1}{n} |\Delta s_{nj}|^{\beta}$$

apply Jensen's inequality

$$\boxed{\beta \leq 1} \overset{\text{Jensen}}{\leq} n \left(\sum_{j=1}^{n} \frac{1}{n} |\Delta s_{nj}| \right)^{\beta}$$

$$= n^{1-\beta} T^{\beta}$$

2 lower

$$\sum_{j=1}^{n} |\Delta s_{nj}|^{\beta} = n \sum_{j=1}^{n} \frac{1}{n} |\Delta s_{nj}|^{\beta}$$

apply Jensen's inequality

$$\boxed{\beta \geq 1} \overset{\text{Jensen}}{\geq} n \left(\sum_{j=1}^{n} \frac{1}{n} |\Delta s_{nj}| \right)^{\beta}$$

$$= n^{1-\beta} T^{\beta}$$

2 upper

$$\sum_{j=1}^{n} |\Delta s_{nj}|^{\beta} = \sum_{j=1}^{n} |\Delta s_{n,j}| \cdot \underbrace{|\Delta s_{nj}|^{\beta-1}}_{\leq \delta_n^{\beta-1}}$$ since $\beta \geq 1$

$$\leq \delta_n^{\beta-1} \sum_{j=1}^{n} |\Delta s_{nj}| = \delta_n^{\beta-1} T$$

3 lower:

$$\left(\sum_{j=1}^{n} |\Delta s_{nj}|^{\beta}\right)^2 = n^2 \left(\sum_{j=1}^{n} \frac{1}{n} |\Delta s_{nj}|^{\beta}\right)^2$$

by Cauchy-Schwarz
$$\leq n^2 \left(\sum_{j=1}^{n} \frac{1}{n} |\Delta s_{nj}|^{2\beta}\right)$$

3 upper

$$\frac{\sum_{j=1}^{n} |\Delta s_{nj}|^{2\beta}}{\left(\sum_{j=1}^{n} |\Delta s_{nj}|^{\beta}\right)^2} = \underbrace{\frac{\sum_{j=1}^{n} |s_{nj}|^{2\beta}}{\sum_{j=1}^{n} |s_{nj}|^{\beta}}}_{\leq \delta_n^{\beta}} \cdot \underbrace{\frac{1}{\sum_{j=1}^{n} |s_{nj}|^{\beta}}}_{\leq \delta_n^{1-\beta} T^{-1}}$$

term-wise

by 1 lower

$\boxed{\square \text{ Lemma}}$

Next, compute $E(V_{n,\alpha})$ and $Var(V_{n,\alpha})$

note: $\Delta B_{nj} \sim \sqrt{\Delta s_{nj}} \cdot \xi$, where $\xi \sim N(0,1)$

Hence $\mathbb{E}\left(|\Delta B_{nj}|^{\beta}\right) = |\Delta s_{nj}|^{\beta/2} \underbrace{E|\xi|^{\beta}}$

notation m_β

$$\mathbb{E}(V_{n,\alpha}) = \mathbb{E}\left(\sum_{j=1}^{n} |\Delta B_{nj}|^{\alpha}\right) =$$

$$= \sum_{j=1}^{n} \mathbb{E}\left(|\Delta B_{nj}|^{\alpha}\right) = m_{\alpha} \sum_{j=1}^{n} |\Delta s_{nj}|^{\frac{\alpha}{2}}$$

$$\mathbb{E}\left((V_{n\alpha})^2\right) = \mathbb{E}\left(\left(\sum_{j=1}^{n} |\Delta B_{nj}|^{\alpha}\right)^2\right) = \dots =$$

$$\sum_{j=1}^{n} \mathbb{E}\left(|\Delta B_{nj}|^{2\alpha}\right) + \sum_{\substack{j,k=1 \\ j\neq k}}^{n} \mathbb{E}\left(|\Delta B_{nj}|^{\alpha} |\Delta B_{nk}|^{\alpha}\right) = \dots$$

independent increments

$$= m_{2\alpha} \sum_{j=1}^{n} |\Delta s_{nj}|^{\alpha} + (m_{\alpha})^2 \sum_{\substack{j,k=1 \\ j\neq k}}^{n} |\Delta s_{nj}|^{\frac{\alpha}{2}} |\Delta s_{nk}|^{\frac{\alpha}{2}}$$

$$= \left(m_{2\alpha} - m_{\alpha}^2\right) \sum_{j=1}^{n} |\Delta s_{nj}|^{\alpha} + \underbrace{m_{\alpha}^2 \left(\sum_{j=1}^{n} |\Delta s_{nj}|^{\frac{\alpha}{2}}\right)^2}_{\left(\mathbb{E}(V_{n,\alpha})\right)^2}$$

$$Var(V_{n,\alpha}) = \mathbb{E}(V_{n\alpha}^2) - \mathbb{E}(V_{n,\alpha})^2$$

$$= (m_{2\alpha} - m_\alpha)^2 \sum_{j=1}^{n} |\Delta s_{n,j}|^\alpha$$

Proof of the Theorem

$\boxed{\alpha = 2}$ $\quad m_2 = 1$

$$\mathbb{E}(V_{n,2}) = m_2 \sum_{j=1}^{n} |\Delta s_{n,j}| = T$$

$$Var(V_{n,2}) = (m_4 - m_2) \sum_{j=1}^{n} |\Delta s_{n,j}|^2$$

by ② upper $\leq (m_4 - m_2)\, T\, \delta_n \longrightarrow 0$

Hence (by Chebyshev) $V_{n,2} \xrightarrow{P} T$

$\boxed{\alpha > 2}$ (note $V_{n,\alpha} \geq 0$)

$$\mathbb{E}(|V_{n,\alpha}|) = \mathbb{E}(V_{n,\alpha}) = m_\alpha \sum_{j=1}^{n} |\Delta s_{n,j}|^{\frac{\alpha}{2}} \leq \quad (\tfrac{\alpha}{2}\ \ 1)$$

$$\leq m_\alpha \cdot T \cdot \delta_n^{\frac{\alpha}{2}-1} \longrightarrow 0$$

Hence, by Markov's inequality $V_{n,\alpha} \xrightarrow{P} 0$

$\boxed{\alpha < 2}$

note: $\frac{\alpha}{2} < 1$

$$E(V_{n,\alpha}) = m_\alpha \sum_{j=1}^{n} |\Delta s_{nj}|^{\frac{\alpha}{2}} \geqslant$$

$$\geqslant T m_\alpha \cdot \delta_n^{\frac{\alpha}{2}-1} \longrightarrow \infty$$

$$\frac{\mathrm{Var}(V_{n,\alpha})}{\mathbb{E}(V_{n,\alpha})^2} = \frac{(m_{2\alpha} - m_\alpha)^2}{m_\alpha^2} \cdot \frac{\sum_{j=1}^{n} |\Delta s_{nj}|^{\alpha}}{\left(\sum_{j=1}^{n} |\Delta s_{nj}|^{\frac{\alpha}{2}}\right)^2} \leqslant$$

by 3 upper

$$\leqslant \frac{(m_{2\alpha} - m_\alpha)^2}{T m_\alpha^2} \delta_n \longrightarrow 0$$

$$\mathbb{P}(V_{n,\alpha} \leqslant M) = \mathbb{P}\left(V_{n\alpha} - \mathbb{E}(V_{n,\alpha}) \leqslant M - \mathbb{E}V_{n,\alpha}\right)$$

$\boxed{\text{n large: } M - \mathbb{E}V_{n,\alpha} < 0}$

$$\leqslant \mathbb{P}\left(|V_{n\alpha} - \mathbb{E}(V_{n,\alpha})| > |\mathbb{E}V_{n\alpha} - M|\right)$$

by Chebyshev

$$\leqslant \frac{\mathrm{Var}(V_{n,\alpha})}{(\mathbb{E}(V_{n,\alpha}) - M)^2} \longrightarrow 0$$

$\square$

Complement on

Quadratic variation of BM.

[stronger convergence]

Let $0 \leq s < t$ be fixed

and

deterministic, not depending on the BM

$$s = t^n_0 < t^n_1 < \cdots < t^n_n = t$$

a sequence of partitions of $[s,t]$

such that

$$\delta_n := \max_{1 \leq j \leq n} |t^n_j - t^n_{j-1}| \to 0, \text{ as } n \to \infty.$$

$$Q_n([s,t]) := \sum_{j=1}^{n} \left|B(t_j) - B(t^n_{j-1})\right|^2$$

the (approximate) quadratic variation on the partition.

Thm $\forall \lambda \in \mathbb{R}$

$$\lim_{n \to \infty} \mathbb{E}\left(e^{\lambda Q_n[s,t]}\right) = e^{\lambda(t-s)}$$

Complement on QV

argument:
$|Y|^{2k} \leq (2k!) \cosh Y$

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Corollary $\forall p \in \mathbb{N}$

$$\lim_{n\to\infty} \mathbb{E}\left(\left|Q_n[s,t] - (t-s)\right|^p\right) = 0$$

(That is: $Q_n[s,t] \xrightarrow{\mathcal{L}^p} (t-s)$.)

Remarks:

① Note that the sequence of partitions was given deterministically — not depending on the realization of $B(\cdot)$.

The problem becomes much more subtle if we let the partitions depend on $B(\cdot)$....

② Actually $Q_n[s,t] \xrightarrow{a.s.} (t-s)$ also holds.

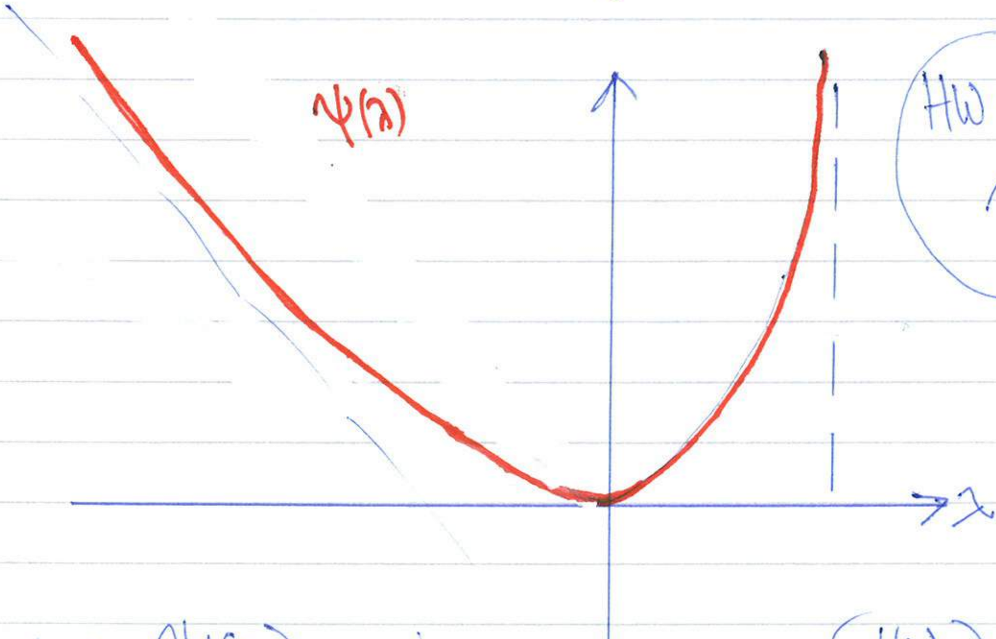
Complement on Q.V

Proof Let $\psi : (-\infty, 1) \longrightarrow \mathbb{R}_+$

$$\psi(\lambda) := \log \mathbb{E}\left(\exp\left\{\frac{\lambda}{2}(\xi^2 - 1)\right\}\right)$$

$\xi \sim \mathcal{N}(0,1)$

$$\psi(\lambda) = -\frac{1}{2}\left(\log(1-\lambda) + \lambda\right)$$



HW: analyze $\psi(\lambda)$

$\lambda \mapsto \psi(\lambda)$ is convex (HW)

for $|\lambda| < \frac{1}{2}$: $\psi(\lambda) \leq C^2 \cdot \lambda^2$ (with some C) HW

Complement on Q.V

$$\mathbb{E}\left(\exp\left\{\lambda\left(Q_n[s,t]-(t-s)\right)\right\}\right) =$$

$$\mathbb{E}\left(\exp\left\{\lambda\sum_{j=1}^{n}\left(\left(B(t_j^n)-B(t_{j-1}^n)\right)^2-(t_j^n-t_{j-1}^n)\right)\right\}\right) \overset{1}{=}$$

$$\prod_{j=1}^{n}\mathbb{E}\left(\exp\lambda\left(\left(B(t_j^n)-B(t_{j-1}^n)\right)^2-(t_j^n-t_{j-1}^n)\right)\right) =$$

$$\prod_{j=1}^{n}\mathbb{E}\left(\exp\lambda(t_j^n-t_{j-1}^n)(\xi^2-1)\right) \overset{2}{=}$$

$$\exp\sum_{j=1}^{n}\psi\left(\lambda(t_j^n-t_{j-1}^n)\right) \overset{3}{\leq}$$

$$\exp\left\{C^2\lambda^2\sum_{j=1}^{n}(t_j^n-t_{j-1}^n)^2\right\} \longrightarrow 1.$$

1: Independent increments

2: scaling

3: quadratic bound on ψ / valid for large n. $\square$