Bálint Tóth:

Filtrations, Stopping times, Martingales and all that (recap)

Recall all you know of measure theory ... including (the general notion of) conditional expectation.

$(\Omega, \mathcal{F}, \mathbb{P})$ a probability space

Filtration: $(\mathcal{F}_t)_{t\geq 0}$ an increasing (nested) collection of σ-algebras:

$$\mathcal{F}_t \subseteq \mathcal{F} \quad \& \quad 0 \leq s \leq t: \mathcal{F}_s \subseteq \mathcal{F}_t.$$

For technical reasons we assume continuity from right:

$$\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$$

Adapted process:

$t \mapsto X(t)$ stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$
is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$
iff $(\forall t)$: $X(t)$ is $\mathcal{F}_t$-measurable.

Natural - own - filtration of the stochastic process $t \mapsto X(\ )$:

$$\mathcal{F}_t^{X,0} := \sigma\left(X(s) : \quad s \leq t\right)$$

made continuous from right:

$$\mathcal{F}_t^{X} := \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}^{X,0}$$ (slight difference...)

Remark

In some cases we may have to care about more than one filtrations.

Stopping times: w.r.t. the filtration $(\mathcal{F}_t)_{t \geq 0}$

$$T : \Omega \longrightarrow \mathbb{R}_+ \quad \text{a random time}$$

such that:

$$(\forall t) : \quad \{T \leq t\} \in \mathcal{F}_t$$

i.e.: with the information available up to time t one can decide whether T occured or not (no need of oracle)

Examples:

$t \longmapsto X(t) \in \mathbb{R}^d$ stoch. proces with continuous sample path (e.g. BM)

$A \subseteq \mathbb{R}^d$ closed set

$$T_A := \inf\{ s > 0 : X(s) \in A \}$$

first hitting time of A is a stopping time

$$L_A := \sup\{ s < 1 : X(s) \in A \}$$

./.

last exit time from A before $t = 1$ is not a stopping time

Markov processes:

$t \longmapsto X(t)$ stochastic process

$\mathcal{F}_t^X := \sigma\{X(s) : s \leq t\}$

its own natural filtration

Markov property:

$t \longmapsto X(t) \in \mathbb{R}^d$ is $\boxed{\text{Markov process}}$

iff for any t_0 and $u \geq 0$ and any

$F : \mathbb{R}^d \longrightarrow \mathbb{R}$ bounded & measurable

$$\mathbb{E}\left(F(X(t_0+u)) \mid \mathcal{F}_{t_0}^X\right) = \mathbb{E}\left(F(X(t_0+u)) \mid \sigma(X(t_0))\right)$$

In plain words:

$$\begin{pmatrix} \text{conditional distrib of } X(t_0+u) \\ \text{given } \{X(s) : s \le t_0\} \end{pmatrix} = \begin{pmatrix} \text{conditional distrib of } X(t_0+u), \\ \text{given } X(t_0) \end{pmatrix}$$

. It follows that: for

$t_0 \le t_1 \le t_2 \le \dots \le t_n$

$F : (\mathbb{R}^d)^n \to \mathbb{R}$ measurable & bdd

$$\mathbb{E}\left(F(X(t_1), X(t_2), \dots, X(t_n)) \,\middle|\, \mathcal{F}^X_{t_0} \right) =$$

$$\mathbb{E}\left(F(X(t_1), X(t_2), \dots, X(t_n)) \,\middle|\, \sigma\left(X(t_0)\right)\right)$$

by applying "tower rule" (successive conditioning) "given the present position (at t_0) past & future are independent"

Example: BM is a Markov process

Strong Markov property

$t \mapsto X(t)$ stochastic process.

$\left(\mathcal{F}_t^X\right)_{t \geq 0}$ its ~~own~~ own natural filtration

T : stopping time w.r.t. $\left(\mathcal{F}_t^X\right)_{t \geq 0}$

$$\mathcal{F}_T := \sigma\left\{ X(s \wedge T) : s \geq 0 \right\}$$

or: $\left(A \in \mathcal{F}_T\right) \Longleftrightarrow \left((\forall t)\, A \cap \{T \leq t\} \in \mathcal{F}_t\right)$

The process $t \mapsto X(t)$ is strongly Markov, iff for any stopping time T, and $u \geq 0$

$$\mathbb{E}\left(F(X(T+u)) \mid \mathcal{F}_T^X \right) = \mathbb{E}\left(F(X(T+u)) \mid \sigma(X(T)) \right).$$

i.e. The Markov property holds for all stopping times, too / not just for deterministic times.

(Strong Markov) is indeed stronger than (Markov)

subtle

Example BM is strongly Markov

A counterexample:

$\mathbb{P}(X(t) \equiv 0) = 1/2$ $\qquad \tau := \inf\{s : X(s) = 0\}$

$\mathbb{P}(X(t) = B(t) + 1) = 1/2$

Martingales:

$(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ filtered probab. space

$t \mapsto X(t) \in \mathbb{R}$ stoch. process adapted to $(\mathcal{F}_t)_{t\geq 0}$

real valued

$t \mapsto X(t)$ is a $\left\{\begin{array}{l} \text{martingale} \\ \text{supermartingale} \\ \text{submartingale} \end{array}\right\}$

iff:

1. $(\forall t)$ $\quad \mathbb{E}(|X(t)|) < \infty$

2. $\forall\, s \leq t$:

for martingal

$$\mathbb{E}(X(t) \mid \mathcal{F}_s) \left\{\begin{array}{c} = \\ \leq \\ \geq \end{array}\right\} X(s)$$

for supermartingale $\qquad$ for submartingale

Examples:

- $B(t)$ is a martingale
- $B(t)^2$ is a submartingale
- More generally:
  if $t \mapsto M(t)$ is a martingale
  $\psi : \mathbb{R} \to \mathbb{R}$ convex and
  $$\mathbb{E}\left(|\psi(M(t))|\right) < \infty$$
  then $Y(t) := \psi(M(t))$ is
  submartingale.

$$\mathbb{E}\left(\psi(M(t)) \,|\, \mathcal{F}_s\right) \geq \text{Jensen's ineq.}$$
$$\psi\left(\mathbb{E}(M(t) \,|\, \mathcal{F}_s)\right) = \psi(M(s))$$

- $B(t)$: BM. ; $\theta \in \mathbb{R}$ fixed parameter

$$t \longmapsto \exp\left\{\theta B(t) - \frac{\theta^2}{2} t\right\} =: M_\theta(t)$$

is a martingale w.r.t $\left(\mathcal{F}_t^B\right)_{t \geq 0}$

HW: check!

Hence
$$\left.\frac{d^n}{d\theta^n} M_\theta(t)\right|_{\theta=0}, \quad n = 1, 2, 3, \dots$$

are martingales HW: check

$B(t)$, $B(t)^2 - t$, $B(t)^3 - 3tB(t)$,

.... are all martingales.

We'll see many more!

In this course we'll deal only with $\boxed{\text{continuous}}$ martingales.

Martingales and stopping times:

$\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ filtered probability sp.

$t \longmapsto X(t)$ martingale, adapted to $(\mathcal{F}_t)_t$
(or sub.., super...)

T: stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$

The stopped martingale:

$$X^T(t) := X(t \wedge T)$$

Thm If $t \longmapsto X(t)$ is a (sub.., super...) martingale with $\boxed{\text{continuous}}$ path then so is $t \longmapsto X^T(t)$.

Optional stopping Theorem:

Let $t \mapsto X(t)$ be a $\boxed{\text{continuous}}$ (sub.. / super..) martingale. If either one of conditions ⓐ, ⓑ or ⓒ below holds, than

$$\mathbb{E}(X(T)) = \mathbb{E}(X(0))$$

$\geq$ for sub..
$\leq$ for super.

ⓐ $(\exists K < \infty)$: $T \leq K$ a.s.
(the stopping time is a.s. bounded)

ⓑ $(\exists K < \infty)$: $(\forall t)$: $|X(t \wedge T)| \leq K$
and $T < \infty$ a.s.
(the stopped martingale is a.s. bounded)

ⓒ $(\exists K < \infty)$: $(\forall t \ \forall h \leq 1)$:

$$\mathbb{E}\left(|X(t+h) - X(t)| \,\middle|\, \mathcal{F}_t\right) \leq K$$

and $\mathbb{E}(T) < \infty$.

Theorem Submartingale or maximal inequality

Let $t \mapsto X(t)$ be a $\boxed{\text{continuous}}$ $\boxed{\text{submartingale}}$ such that $(\forall t): X(t) \geq 0$ a.s.

Then:

$$P\left(\max_{0 \leq s \leq t} X(s) \geq a\right) \leq a^{-1} E(X(t)).$$

Remark Recall Chebyshev's and Kolmogorov's inequalities.

Proofs: Due to continuity of $t \mapsto X(t)$, can be reduced to the corresponding discrete time theorems, by

$X_k^{(n)} := X\left(\frac{k}{n}\right)$, $\mathcal{F}_k^{(n)} := \mathcal{F}_{\frac{k}{n}}$, $T^{(n)} =$

$T^{(n)} := n^{-1} \lceil nT \rceil$ ...