Bálint Tóth: Itô calculus /2

Itô's formula

$\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ filtered probability space

$B(t)$ BM, $\mathcal{F}_t^B \subseteq \mathcal{F}_t$, $\mathcal{F}_t$-martingale

Itô processes: (def).

$$X(t) = X(0) + \int_0^t u(s)\,ds + \int_0^t v(s)\,dB(s)$$

(i) $u(\cdot), v(\cdot)$ progressively measurable

(ii) $\mathbb{E}\left(\int_0^T |u(s)|\,ds\right) < \infty$

(iii) $\mathbb{E}\left(\int_0^T |v(s)|^2\,ds\right) < \infty$ $\lrcorner$

Notation:

$$dX(t) = u(t)\,dt + v(t)\,dB(t)$$

Theorem: Let $g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be $C^{1,2}$; $X(t)$ Itô process with

$$dX(t) = u(t)\,dt + v(t)\,dB(t);$$

$$Y(t) := g(t, X(t)).$$

Then $Y(t)$ is also Itô process with

$$dY(t) = \Big(\dot{g}(t,X(t)) + g'(t,X(t))\,u(t) + \frac{1}{2} g''(t,X(t))\,v(t)^2\Big)\,dt + g'(t,X(t))\,v(t)\,dB(t)$$

$$= \dot{g}(t,X(t))\,dt + g'(t,X(t))\,dX(t) + \frac{1}{2} g''(t,X(t))\,v(t)^2\,dt$$

Cook-book:

$$(dB(t))^2 = dt, \quad (dB(t))^3 = dt\,dB(t) = (dt)^2 = 0$$

$$(dX(t))^2 = \sigma(t)^2 dt, \quad (dX(t))^3 = dt\,dX(t) = (dt)^2 = 0$$

Examples: $X(t) = B(t) \quad (\mu \equiv 0, \ \sigma \equiv 1)$

① $g(t,x) = g(x) \qquad g \in C^2$

$$g(B(t)) - g(B(0)) = \int_0^t g'(B(s))\,dB(s) + \frac{1}{2}\int_0^t g''(B(s))\,ds$$

$$\int_0^t f(B(s))\,dB(s) = F(B(t)) - F(B(0)) - \frac{1}{2}\int_0^t f'(B(s))\,ds$$

$f \in C^1$

(2) $g(t,x) = x\, g(t)$ $\qquad g \in C^1$

$$\int g(s)\,dB(s) = g(t)B(t) - \int_0^t \dot{g}(s)\,B(s)\,ds$$

(could be derived directly by integration by parts)

Full proof for $g(t,x) = g(x)$ at end of this set of notes

Proof of Itô's formula (sketch)

assume for simplicity $g \in C^{2,3}$

$$t_j = t \cdot \frac{j}{n} \qquad j = 0, 1, \dots, n$$

$$Y(t) - Y(0) = \sum_{j=1}^{n} Y(t_j) - Y(t_{j-1})$$

$$Y(t_j) - Y(t_{j-1}) = g(t_j, X(t_j)) - g(t_{j-1}, X(t_{j-1})) \underset{\text{expansion}}{\overset{\text{Taylor}}{=}}$$

$$\dot{g}(t_{j-1}, X(t_{j-1}))(t_j - t_{j-1}) +$$

$$g'(t_{j-1}, X(t_{j-1}))\big(u(t_j)(t_j - t_{j-1}) + v(t_{j-1})(B(t_j) - B(t_{j-1}))\big)$$

$+ \%$

$$+\frac{1}{2}\, g''(t_{j-1}, X(t_{j-1}))\, v(t_{j-1})^2 \,(B(t_j)-B(t_{j-1}))^2$$

$+ \text{error}_j$

$$|\text{error}_j| \leq C\left((\underset{\substack{\uparrow \\ t_j - t_{j-1}}}{\Delta t_j})^2 + |\underset{\substack{\uparrow \\ B(t_j)-B(t_{j-1})}}{\Delta B_j}|^3 \right)$$

$$\sum_j |\text{error}_j| \leq C \sum_j \left(|\Delta t_j|^2 + |B_j|^3 \right) \longrightarrow 0$$

as $n \to \infty$

this is L^2-limit

$$Y(t) - Y(0) =$$

$$\lim_{n\to\infty} \sum_j \Big\{ \dot{g}_j (\Delta t_j) + g'_j \left(u_j (\Delta t_j) + v_j \Delta B_j \right) +$$

$$\frac{1}{2} g''_j\, v_j^2 (\Delta t_j) \Big\} =$$

$$\int_0^t \left(\dot{g}(s, X(s)) + g'(s, X(s))\, u(s) + \frac{1}{2} g''(s, X(s))\, v(s)^2 \right) ds$$

$$+ \int_0^t g'(s, X(s))\, v(s)\, dB(s)$$

$\square$

Multidimensional Itô formula:

$B = (B_1, B_2, \dots, B_m)^T$ Brownian motion in $\mathbb{R}^m$

$u = (u_1, u_2, \dots, u_n)^T$ process in $\mathbb{R}^n$

$v = (v_{ji})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ process in $\mathbb{R}^{n\times m}$

$$dX(t) = u(t)\,dt + v(t)\,dB(t)$$

$$dX_j(t) = u_j(t)\,dt + v_{ji}(t)\,dB_i(t)$$

$$g : [0,\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^p, \quad C^{1,2}$$

$$g(t,x) = \left(g_k(t, x_1, \dots, x_n)\right)_{k=1,\dots,p}$$

$$Y(t) := g(t, X(t)).$$

Thm Multidimensional Itô formula.

$$dY_k(t) =$$

$$\left(\frac{\partial g_k}{\partial t}(t, X(t)) + \sum_{j=1}^{n} \frac{\partial g_k}{\partial x_j}(t, X(t))\, u_j(t) + \right.$$

$$\left. + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, X(t)) \sum_{r=1}^{m} v_{ir}(t)\, v_{jr}(t) \right) dt$$

$$+ \sum_{j=1}^{n} \sum_{r=1}^{m} \frac{\partial g_k}{\partial x_j}(t, X(t))\, v_{jr}(t)\, dB_r(t) .$$

mind the indices!

Applications of Itô: see exercises

Representation of random variables and martingales as Itô integrals:

$(\Omega, \mathcal{F}, \mathbb{P})$ $\quad t \mapsto B(t)$ $\quad$ BM

now this is the filtration considered !!! $\longrightarrow$ $(\mathcal{F}_t^B)_{t \geq 0}$ natural filtration of B.

Theorem (Itô representation Thm)

$$\{X \in L^2(\Omega, \mathcal{F}_t^B, \mathbb{P})\}$$
$$\|$$
$$\left\{c + \int_0^t \varphi(s)\, dB(s) : c = \mathbb{E}(X) \in \mathbb{R};\ \varphi \in \mathcal{V}_T\right\}$$

Theorem (martingale representation thm)

Let $M(t)$ be a $(\mathcal{F}_t^B)$-martingale, L^2

Then $\exists!\ \varphi \in \mathcal{V}_T$:

$$M(t) = \int_0^T \varphi(s)\, dB(s)$$

$h : [0,\infty) \to \mathbb{R}$ deterministic s.t.

$$(\forall t < \infty) : \int_0^t |h(s)|^2 ds < \infty$$

$$Y_h(t) := \exp\left(\int_0^t h(s)\, dB(s) - \frac{1}{2} \int_0^t h(s)^2 ds \right)$$

Lemma 1

$$Y_h(t) = 1 + \int_0^t Y_h(s) h(s)\, dB(s)$$

Proof apply Ito with $dX(t) = h(s) dB(s) - \frac{1}{2} h(s)^2 ds$

$$g(t,x) = e^x$$

get $dY_h(t) = Y_h(s) h(s)\, dB(s)$ $\square$

Note : $t \mapsto Y_h(t)$ is a martingale!

Lemma 2 :

linear span of $\{ Y_h(t) : h \text{ locally bdd} \}$ is dense in $L^2(\Omega, \mathcal{F}_t^B, \mathbb{P})$.

Proof : tedious but no surprizes

Itô repr. thm follows from Lemma 1 + Lemma 2 $\blacksquare$

Proof of martingale representation thm:

$M(t)$ be $\left(\mathcal{F}_t^B\right)$ - martingale

by Itô repres. thm: $(\forall t)$ $M(t) = \int_0^t \varphi^t(s)\,dB(s)$

mind dependence on t

Let $0 \le t_1 \le t_2$

$$\mathbb{E}\left(M(t_2) \,\middle|\, \mathcal{F}_{t_1}^B\right) = \mathbb{E}\left(\int_0^{t_2} \varphi^{t_2}(s)\,dB(s) \,\middle|\, \mathcal{F}_{t_1}^B\right) = \int_0^{t_1} \varphi^{t_2}(s)\,dB(s)$$

$$\| $$

$$M(t_1) = \int_0^{t_1} \varphi^{t_1}(s)\,dB(s)$$

It follows that:

$$\mathbb{E}\left(\int_0^{t_1} \left(\varphi^{t_2}(s) - \varphi^{t_1}(s)\right)^2 ds\right) = 0$$

$$\Rightarrow \varphi^{t_2}(s) = \varphi^{t_1}(s) \quad (s,\omega)\text{-a.e.} \quad \square$$

Some details Proof of Ito's formula: assume $|f'''|$ bdd

$$f(B(t)) - f(B(0)) = \int_0^t f'(B(s))\,dB(s) + \frac{1}{2}\int_0^t f''(B(s))\,ds$$

$$f(B(t)) - f(B(0)) = \sum_{j=0}^{n-1} \left(f(B(t_{i+1})) - f(B(t_i)) \right) =$$

$$f(B(t_{i+1})) - f(B(t_i)) =$$

$$f'(B(t_i)) \left(B(t_{i+1}) - B(t_i)\right) +$$

$$\left\{ \begin{array}{l} \frac{1}{2} f''(B(t_i)) \left(\left(B(t_{i+1}) - B(t_i)\right)^2 - (t_{i+1} - t_i) \right) + \\ \frac{1}{2} f''(B(t_i)) (t_{i+1} - t_i) \end{array} \right. +$$

$$(\text{Taylor error})_j$$

Taylor error_j =

$$\frac{1}{6} \cdot f'''(B(t_j^*))\,(B(t_{j+1})-B(t_j))^3$$

$$\boxed{E\left(\left(\sum_{j=0}^{n-1} \text{Taylor error}_j\right)^2\right) \longrightarrow 0 \text{ as } n \to \infty}$$

$$E\left(\left(\sum_{j=0}^{n-1} f''(B(t_j))\left((B(t_{j+1})-B(t_j))^2-(t_{j+1}-t_j)\right)\right)^2\right)$$

$$=\sum_{j=0}^{n-1} E\left(\left(f''(B(t_j))\right)^2\right)(\Delta t_j)^2\ (m_4-m_2)$$

$$\longrightarrow 0$$

$$\Rightarrow E\left(\left(\sum_{j=0}^{n-1} f'''(B(t_j^*))\,(B(t_{j+1})-B(t_j))^3\right)^2\right)$$

$$< \mathbb{E}\left(\sum_{j=0}^{n-1} \left|f'''(B(t_j^*))\right|^2 \Delta t_j\right) \cdot \mathbb{E}\left(\sum_{j=0}^{n-1} \frac{\left|\Delta B_j\right|^6}{\underbrace{\Delta t_j}_{(\Delta t_j)^2}}\right)$$

$$f(t, B(t)) - f(0, B(0)) = \int_0^t \left(\dot{f}(s, B(s)) + \frac{1}{2} f''(s, B(s)) \right) ds + \int_0^t f'(B(s))\, dB(s)$$