Diffusions/1

Balint Toth:

Infinitesimal generator, Dynkin's formula, Kolmogorov's backward equation

We consider now the time-homogeneous case:

$b : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad |b(x)-b(y)| \leq C|x-y|$

$\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}, \quad |\sigma(x)-\sigma(y)| \leq C|x-y|$

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))\,dB(t) \\ X(0) = x \end{cases}$$

The (unique) strong solution is a homogeneous strong Markov process (we don't prove it here)

A useful lemma about Ito processes and stopping times:

Ingredients: $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$

$t \mapsto B(t) \in \mathbb{R}^m$ BM + $(\mathcal{F}_t)$-martingale

$$\left.\begin{aligned} s &\mapsto u(s) \in \mathbb{R}^n \\ s &\mapsto v(s) \in \mathbb{R}^{n\times m} \end{aligned}\right\} \text{progressively measurable}$$

$$Y(t) = y + \int_0^t u(s)\,ds + \int_0^t v(s)\,dB(s) \; \Big| \in \mathbb{R}^n$$

$f: \mathbb{R}^n \to \mathbb{R}$, C_0^2 (of compact support)

assume ⊛ $\left(|u(s)| + |v(s)|\right)\mathbb{1}\left(Y(s) \in \operatorname{supp} f\right)$ be bounded

τ: $(\mathcal{F}_t)$-stopping time, $E(\tau) < \infty$.

Lemma: With the notation and conditions as stated:

$$\mathbb{E}\left(f(Y(\tau))\right) = f(y) + \mathbb{E}\left(\int_0^\tau \sum_i \mu_i(s) \frac{\partial f}{\partial x_i}(Y(s))\, ds + \frac{1}{2}\int_0^\tau \sum_{ij} (\nu \cdot \nu^T)_{ij}(s) \frac{\partial^2 f}{\partial x_i \partial x_j}(Y(s))\, ds\right)$$

Proof Itô's formula + optional stopping

$$Z(s) := f(Y(s))$$

$$dZ(s) = \left(\sum_i \frac{\partial f}{\partial x_i}(Y(s))\mu_i(s) + \frac{1}{2}\sum_{ij}(\nu \cdot \nu^T)_{ij}(s)\frac{\partial^2 f}{\partial x_i \partial x_j}(Y(s))\right) ds$$

$$+ \sum_{ik} \frac{\partial f}{\partial x_i}(Y(s))\, \nu_{ik}(s)\, dB_k(s)$$

$$f(Y(t)) = f(y) +$$

$$\int_0^t \left\{ \frac{\partial f}{\partial x_i}(Y(s))\, u_i(s) + \frac{1}{2} \frac{\partial^2 f(Y(s))}{\partial x_i \partial x_j} (v v^T)_{ij}(s) \right\} ds +$$

$$+ \underbrace{\int_0^t \sum_{i,k} \frac{\partial f}{\partial x_i}(Y(s))\, v_{ik}(s)\, dB_k(s)}_{\text{this is a martingale}}$$

OST can be applied. $\square$

Remark: Condition $(*)$ (see page 2.) needed in order that ① all expectations be well defined

② OST be applicable.

The infinitesimal generator:

Recall: infinit. gen. of cont. time finite MC-s

5

$$\begin{cases} dX(t) = b(X(t))\,dt + \sigma(X(t))\,dB(t) \\ X(0) = x \end{cases}$$

Notation $\mathbb{E}_x(\dots)$ = expectation w.r.t. distribution of this process.

$$\mathcal{D}_A := \left\{ f \in C_0(\mathbb{R}^n) : \lim_{h \downarrow 0} \frac{\mathbb{E}_x(f(X(h))) - f(x)}{h} \text{ exists} \right\}$$

$$f \in \mathcal{D}_A \quad : \quad Af(x) = \lim_{h \downarrow 0} \frac{\mathbb{E}_x(f(X(h))) - f(x)}{h}$$

$\mathcal{D}_A$ = the domain of definition of the infinitesimal generator

A = the infinitesimal generator

Theorem $C_0^2(\mathbb{R}^n) \subset \mathcal{D}_A$ and for $f \in C_0^2(\mathbb{R}^n)$

$$Af(x) = \sum_i b_i(x)\partial_i f(x) + \frac{1}{2}\sum_{ij}(\sigma\sigma^T)_{ij}(x)\,\partial_{ij}^2 f(x)$$

Proof follows directly from the Lemma:

Notation for future use

$\sigma \cdot \sigma^T =: a$

positive semidefinite

$$\mathbb{E}_x\big(f(X(h))\big) - f(x) =$$

$$\mathbb{E}_x\Big(\Big(\int_0^h \sum_i \frac{\partial f}{\partial x_i}(X(s))\, b_i(X(s)) + \frac{1}{2}\sum_{ij}\frac{\partial^2 f(X(s))}{\partial x_i \partial x_j}(\sigma\sigma^T)_{ij}(X(s))\Big) ds\Big) \overset{\text{Taylor}}{=}$$

$$\Big(\sum_i b_i(x)\partial_i f(x) + \frac{1}{2}\sum_{ij}(\sigma\sigma^T)_{ij}(x)\,\partial_{ij}^2 f(x)\Big)h + o(h)$$

$\square$

Thm (Dynkin's formula):

$f \in C_0^2(\mathbb{R}^n)$; τ stopping time, $\mathbb{E}(\tau) < \infty$

$$\mathbb{E}_x\big(f(X(\tau))\big) = f(x) + \mathbb{E}_x\left(\int_0^\tau Af(X(s))\,ds\right).$$

Proof: follows directly from the Lemma $\square$

Examples for infinitesimal generators

① Brownian Motion in $\mathbb{R}^n$

$$A = \frac{1}{2}\Delta$$

② Ornstein-Uhlenbeck process

$$dV(t) = -\gamma V(t)\,dt + \sigma\, dB(t)$$

$$A = -\gamma x \frac{\partial}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}$$

③ Geometric Brownian Motion

$$dX(t) = r X(t)\,dt + a X(t)\, dB(t)$$

$$A = r x \frac{\partial}{\partial x} + \frac{a^2}{2} \cdot x^2 \frac{\partial^2}{\partial x^2}$$

④ Fisher-Wright process

$$dX(t) = \sqrt{X(t)(1-X(t))}\, dB(t)\ ; \quad 0 \leq X(t) \leq 1.$$

$$A = \frac{1}{2} X(1-X) \frac{\partial^2}{\partial x^2}$$

[Note: The Lipschitz condition doesn't hold at $x=0$, $x=1$. Care needed!!]

Applications of Dynkin's formula:

1. $t \longmapsto B(t) \in \mathbb{R}^n$, standard BM

$$B(0) = 0$$

$$\tau_R = \inf\{t : |B(t)| = R\}$$

$$\mathbb{E}_0(\tau_R) = ?$$

Choose $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) := |x|^2$

Dynkin:

$$\mathbb{E}_0\Big(\underbrace{f(B(\tau_R))}_{= R^2}\Big) = \underbrace{f(0)}_{= 0} + \mathbb{E}_0\Big(\int_0^{\tau_R} \underbrace{\tfrac{1}{2}\Delta f(B(s))}_{= n}\, ds\Big)$$

$$\mathbb{E}_0(\tau_R) = \frac{R^2}{n}$$

let $x \in \mathbb{R}^n$ $|x| = r < R$

$$\mathbb{E}_x(\tau_R) = ?$$

by strong Markov property and spherical symmetry of BM:

$$\mathbb{E}_0(\tau_R) = \mathbb{E}_0(\tau_r) + \mathbb{E}_x(\tau_R)$$

hence

$$\mathbb{E}_x(\tau_R) = \frac{R^2 - |x|^2}{n} \quad \text{if } |x| \leq R$$

2) Let $0 < r < |x| < R < \infty$

Question: $\mathbb{P}_x(\tau_r < \tau_R) = ?$

Choose f so that:

Laplace eq. with Dirichlet boundary values

$$\Delta f(x) = 0 \quad \text{in } \{x \in \mathbb{R}^n : r < |x| < R\}$$

$$f(x) = \begin{cases} 1 & \text{for } |x| = r \\ 0 & \text{for } |x| = R \end{cases}$$

Dynkin:

$$\mathbb{E}_x\Big(\underbrace{f(B(\tau_r \wedge \tau_R))}_{\mathbb{1}(\tau_r < \tau_R)}\Big) = f(x) + \mathbb{E}_x\Big(\underbrace{\int_0^{\tau_R \wedge \tau_r} \frac{1}{2}\Delta f(B(s))\,ds}_{=0}\Big)$$

Thus: $\mathbb{P}_x(\tau_r < \tau_R) = f(x)$, $r < |x| < R$

Now, solve the PDE boundary value problem:

$\boxed{n = 2}$

$$f(x) = \frac{\ln R - \ln|x|}{\ln R - \ln r}$$ HW: check it

$$\mathbb{P}_x(\tau_r < \tau_R) = \frac{\ln R - \ln|x|}{\ln R - \ln r}$$

$\boxed{n \geq 3}$

$$f(x) = \frac{|x|^{2-n} - R^{2-n}}{r^{2-n} - R^{2-n}}$$

$$\mathbb{P}_x(\tau_r < \tau_R) = \frac{|x|^{2-n} - R^{2-n}}{r^{2-n} - R^{2-n}}$$

Recurrence / Transience of BM in $\mathbb{R}^n$:

Question: Does the BM starting at x, with $|x| > r$ ever hit the ball of radius r, centred at 0?

$$|x| > r, \quad \mathbb{P}_x(\tau_r < \infty) = ?$$

$$\mathbb{P}_x(\tau_r < \infty) = \lim_{R\to\infty} \mathbb{P}_x(\tau_r < \tau_R) =$$

$$\begin{cases} \lim\limits_{R\to\infty} \dfrac{\ln R - \ln|x|}{\ln R - \ln r} = 1, & n = 2 \\[2ex] \lim\limits_{R\to\infty} \dfrac{|x|^{2-n} - R^{2-n}}{r^{2-n} - R^{2-n}} = \left(\dfrac{r}{|x|}\right)^{n-2} < 1, & n \geq 3 \end{cases}$$

2 dimensional Brownian motion is recurrent
3 (and more) dim. Brownian motion is transient