Parabolic, initial value (Cauchy) problem

The heat equation

$u : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$u \in C^{1,2}\big((0,\infty) \times \mathbb{R}^n\big) \cap C\big([0,\infty) \times \mathbb{R}^n\big)$

Heq $\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2} \Delta u(t,x), \text{ for } (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0,x) = f(x) \end{cases}$

where $f \in C(\mathbb{R}^n)$

Thm The unique $C^{1,2}(\mathring{\mathbb{R}}_+ \times \mathbb{R}^n) \cap C(\overline{\mathbb{R}_+} \times \mathbb{R}^n)$ solution of the initial value problem Heq [stands for Heat-equation] is

$$u(t,x) = \mathbb{E}_x\big(f(B(t))\big)$$

where $t \mapsto B(t)$ is BM in $\mathbb{R}^n$, $B(0) = x$.

Actually : explicit solution

$$u(t,x) = \int_{\mathbb{R}^n} \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|y-x|^2}{2t}} f(y)\, dy$$

More general : Kolmogorov Backward Equation

$$Af(x) = \sum_{i=1}^{n} b_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

with Lipschitz conditions in force

$t \mapsto X(t)$ diffusion with infinitesimal gen. A

KBE
$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = Af(t,x) & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0,x) = f(x) \end{cases}$$

<u>Theorem</u> The unique $C^{1,2}(\mathring{\mathbb{R}}_+ \times \mathbb{R}^n) \cap C(\overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ solution of the initial value problem (KBE) is

$$u(t,x) = \mathbb{E}_x\left(f(X(t))\right).$$

Proof seen already.

No explicit formula - if b, σ not constant

<u>Kolmogorov Forward Equation</u> (KFE)

also seen earlier

The Feynman-Kac formula

Let $V : \mathbb{R}^n \to \mathbb{R}$ be (piecewise) continuous, and bounded from below
"potential"

The Feynman-Kac equation:

$$
\text{F-K} \quad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) - V(x)u(t,x) & \text{for } t > 0 \\ u(0,x) = f(x) \end{cases}
$$

Note: $-\frac{1}{2}\Delta + V = H$
is the quantum mechanical Hamiltonian of a particle in potential V:

$-\frac{1}{2}\Delta$ = kinetic energy operator ; V = potential energy operator

$$\text{(Sch)} \quad -i \frac{\partial \psi}{\partial t}(t,x) = -\frac{1}{2}\Delta \psi(t,x) + V(x)\psi(t,x)$$

is the Schrödinger equation for one q.m. particle in potential V.

Thus: $\boxed{\text{(F-K)} = \text{(Sch)} \ \underline{\text{with imaginary time}}}$

<u>Theorem</u> The unique $C^{1,2}(\mathring{\mathbb{R}}_+ \times \mathbb{R}^n) \cap C_{loc}(\bar{\mathbb{R}}_+ \times \mathbb{R}^n)$ solution of the initial value problem (F-K) is

$$u(t,x) = \mathbb{E}_x\left(e^{-\int_0^t V(B(s))\,ds} f(B(t))\right)$$

where $t \mapsto B(t)$ is BM, $B(0) = x$.

Proof of F-K:

very similar to KBE proof.
Here is the $\boxed{\text{main argument}}$ only

$$u(t+h,x) =$$

$$\mathbb{E}_x\left(e^{-\int_0^{t+h} V(B(s))\,ds} f(B(t+h))\right) =$$

$$\mathbb{E}_x\left(e^{-\int_0^{h} V(B(s))\,ds} \cdot e^{-\int_h^{t+h} V(B(s))\,ds} f(B(t+h))\right) =$$

$$\mathbb{E}_x\left(e^{-\int_0^{h} V(B_s)\,ds}\, \mathbb{E}_{B(h)}\left(e^{-\int_0^{t} V(B(s))\,ds} f(B(t))\right)\right) =$$

$$\mathbb{E}_x\left(e^{-\int_0^{h} V(B(s))\,ds}\, u(t, B(h))\right)$$

use Markov property here

So far: identity. Next: let $h \ll 1$.

$$\mathbb{E}_x\left(e^{-\int_0^h V(B(s))\,ds}\, u(t,B(h))\right) =$$

$$\left(1-hV(x)\right)\mathbb{E}_x\left(u(t,B(h))\right) +$$

$$\mathbb{E}_x\left(\left(e^{-\int_0^h V(B(s))\,ds} - 1 + hV(x)\right) u(t,B(h))\right)$$

① $$\mathbb{E}_x\left(u(t,B(h))\right) = u(t,x) + \frac{h}{2}\Delta u(t,x) + o(h)$$

② $$\mathbb{E}_x\left(\left(e^{-\int_0^h V(B(s))\,ds} - 1 - hV(x)\right) u(t,B(h))\right) = o(h)$$

Altogether:

$$u(t+h,x) = u(t,x) + h\left(\frac{1}{2}\Delta u(t,x) - V(x)\,u(t,x)\right) + o(h)$$

or:

$$\frac{u(t+h,x) - u(t,x)}{h} = \frac{1}{2}\Delta u(t,x) - V(x)\,u(t,x) + o(1).$$

□.

An Application of F-K:

$B(t) =$ 1d BM, $B(0) = x$

$$\zeta(t) = \left|\{s \leq t : B(s) \geq 0\}\right| = \int_0^t \mathbb{1}(B(s) \geq 0)\,ds$$

$$u(t,x) = \mathbb{E}_x\left(e^{-\int_0^t \mathbb{1}(B(s)>0)\,ds}\right)$$

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_x^2 u - \mathbb{1}(x \geq 0)\,u \\ u(0,x) = 1, \quad u(t,-\infty) = 1, \quad u(t,+\infty) = e^{-t} \end{cases}$$

$$\hat{u}(\lambda,x) := \lambda \int_0^\infty e^{-\lambda t} u(t,x)\,dt, \qquad \lambda > 0.$$

$$\begin{cases} -\frac{1}{2}\partial_x^2 \hat{u}(\lambda,x) + (\lambda + \mathbb{1}(x \geq 0))\,\hat{u}(\lambda,x) = -\lambda \\ \hat{u}(\lambda,-\infty) = 1, \quad \hat{u}(\lambda,+\infty) = \frac{\lambda}{1+\lambda} \end{cases}$$

Solution

$\boxed{x<0}$: $\hat{u}(\lambda,x) = 1 + A e^{x\sqrt{2\lambda}} + B e^{-x\sqrt{2\lambda}}$

$\boxed{x>0}$ $\hat{u}(\lambda,x) = \frac{\lambda}{1+\lambda} + C e^{x\sqrt{2(\lambda+1)}} + D e^{-x\sqrt{2(\lambda+1)}}$

u bdd. $\Rightarrow$ $B = C = 0$

$\hat{u}(\lambda, x)$ and $\partial_x \hat{u}(\lambda, x)$ continuous a $x=0$ (to be justified)

$$1 + A = \frac{\lambda}{1+\lambda} + D$$

$$\sqrt{2\lambda}\, A = -\sqrt{2(\lambda+1)}\, D$$

$$A = -\frac{\sqrt{\lambda+1}}{\sqrt{\lambda} + \sqrt{\lambda+1}} \cdot \frac{1}{\lambda+1} \; ; \; D = \frac{\sqrt{\lambda}}{\sqrt{\lambda} + \sqrt{\lambda+1}} \cdot \frac{1}{\lambda+1}$$

$$\hat{u}(\lambda, x=0) = 1 + A = \sqrt{\frac{\lambda}{\lambda+1}}$$

but

$$u(t, x=0) = \mathbb{E}_x\left(e^{-\int_0^t \mathbb{1}(B(s)>0)\,ds}\right)$$

by scaling of BM

$$= \mathbb{E}_x\left(e^{-t\int_0^1 \mathbb{1}(B(s)>0)}\right) =$$

$$= \mathbb{E}_x\left(e^{-t\xi}\right)$$

$$\hat{u}(\lambda, x=0) = \lambda \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left(e^{-t\xi}\right) d\lambda = \dots$$

$$= \mathbb{E}_0\left(\frac{\lambda}{\lambda+\xi}\right) = \mathbb{E}_0\left(\frac{1}{1+\xi/\lambda}\right)$$

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We have derived:

$$\mathbb{E}_0\left(\frac{1}{1+\gamma\xi}\right) = \sqrt{\frac{1}{1+\gamma}} \quad \forall \gamma > 0$$

This identifies the arcsine law:

$$\frac{2}{\pi}\int_0^1 \frac{1}{1+\gamma x}\frac{dx}{\sqrt{x(1-x)}} = \frac{1}{\sqrt{1+\gamma}} \quad \square$$