# Stochastic Differential Equations
# Problem Set 2
# Filtrations, Stopping Times, Markov Property, Martingales, ...

**2.1** Let $t \mapsto X(t)$ be a stochastic process in a complete separable metric space $S$. Prove that the following two formulations of the Markov property are actually equivalent. (Note that formulation (b) is a priori stronger than (a).)

a.) For any $0 \leq t$, $0 \leq u$ and $F : S \to \mathbb{R}$ bounded and measurable

$$\mathbf{E}\big(F(X(t+u)) \mid \mathcal{F}_t^X\big) = \mathbf{E}\big(F(X(t+u)) \mid \sigma(X_t)\big).$$

b.) For any $0 \leq t$, $n \in \mathbb{N}$, $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n$ and $F : S^n \to \mathbb{R}$ bounded and measurable

$$\mathbf{E}\big(F(X(t+u_1), X(t+u_2), \ldots, X(t+u_n)) \mid \mathcal{F}_t^X\big) =$$
$$\mathbf{E}\big(F(X(t+u_1), X(t+u_2), \ldots, X(t+u_n)) \mid \sigma(X_t)\big).$$

*Hint:* Apply the "tower rule" of conditional probabilities.

**2.2** a.) Prove that $t \mapsto B(t)$ is a martingale and $t \mapsto B(t)^2$ is a submartingale (with respect to the filtration $(\mathcal{F}_t^B)_{t\geq 0}$).

b.) Let $t \mapsto M(t)$ be a martingale (w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$) and $\psi : \mathbb{R} \to \mathbb{R}$ a *convex* function. Let

$$Y(t) := \psi(M(t)).$$

Assuming that $\mathbf{E}\left(\lvert\, \psi(M(t)) \,\rvert\right) < \infty$ for all $t \geq 0$, prove that $t \mapsto Y(t)$ is a *submartingale. Hint:* Use Jensen's inequality.

**2.3** Show that the processes $t \mapsto B(t)$, $t \mapsto B(t)^2 - t$ and $t \mapsto B(t)^3 - 3tB(t)$ are martingales adapted to the filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$.

**2.4** Check whether the following processes are martingales with respect to the filtration $\left(\mathcal{F}_t^B\right)_{t\geq 0}$:

$$
\begin{aligned}
&(a) \qquad && X(t) = B(t) + 4t,\\
&(b) \qquad && X(t) = B(t)^2,\\
&(c) \qquad && X(t) = t^2 B(t) - 2\int_0^t sB(s)ds,\\
&(d) \qquad && X(t) = B_1(t)B_2(t),
\end{aligned}
$$

where $B_1$ and $B_2$ are two independent Brownian motions.

**2.5** Let $-a < 0 < b$ and denote

$$
\begin{aligned}
\tau_{\text{left}} &:= \inf\{s > 0 : B(s) = -a\},\\
\tau_{\text{right}} &:= \inf\{s > 0 : B(s) = b\},\\
\tau &:= \min\{\tau_{\text{left}}, \tau_{\text{right}}\}.
\end{aligned}
$$

a.) By applying the Optional Stopping Theorem compute $\mathbf{P}\left(\tau_{\text{left}} < \tau_{\text{right}}\right)$ and $\mathbf{E}\left(\tau\right)$.

b.) By "applying" the Optional Stopping Theorem it would "follow" that $\mathbf{E}\left(B(\tau_a)\right) = 0$. However, clearly $B(\tau_a) = a$ by definition (and continuity of the Brownian motion). What is wrong with the argument?

**2.6** a.) Let $\theta \in \mathbb{R}$ be a fixed parameter. Show that the processes $t \mapsto \exp\{\theta B(t) - \theta^2 t/2\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$.

b.) By differentiating with respect to $\theta$ and letting then $\theta = 0$ derive a martingale which is a fourth order polynomial expression of $B(t)$

c.) For any $n \in \mathbb{N}$ let

$$H_n(x) := e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}.$$

Show that $H_n(x)$ is a polynomial of order $n$ in the variable $x$. (It is called the *Hermite polynomial* of order $n$.). Compute $H_n(x)$ for $n = 1, 2, 3, 4$.

d.) Show that for any $n \in \mathbb{N}$ the process $t \mapsto t^{n/2}H_n(B(t)/\sqrt{t})$ is a martingale.

**2.7** Let $t \mapsto B(t)$ be standard $1d$ Brownian motion and $\tau := \inf\{t > 0 : |B(t)| = 1\}$. Prove that

$$\mathbf{E}\left(e^{-\lambda\tau}\right) = \cosh(\sqrt{2\lambda})^{-1}, \qquad \lambda \geq 0.$$

*Hint:* Apply the Optional Stopping Theorem to the exponential martingale defined in problem 2.6.

**2.8** Denote

$$J : \mathbb{R} \to \mathbb{R}, \qquad J(\lambda) := \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{\lambda \cos\theta} d\theta.$$

Let $B(t) = (B_1(t), B_2(t))$ be a *two-dimensional* Brownian motion and

$$\tau := \inf\{t : |\, B(t)\,| = 1\}.$$

That is: $\tau$ is the first hitting time of the circle centred at the origin, with radius 1. Prove that

$$\mathbf{E}\left(e^{-\lambda\tau}\right) = J(\sqrt{2\lambda})^{-1}, \qquad \lambda \geq 0.$$

*Hint*: Apply the Optional Stopping Theorem to the *martingale* $t \mapsto \exp\{\theta \cdot B(t) - |\,\theta\,|^2 t/2\}$, where $\theta \in \mathbb{R}^2$, with the stopping time $\tau$.

**2.9** Let $B(t)$ be a standard Brownian motion and let $\xi$ be a random variable with *Bernoulli* $\left(\frac{1}{2}\right)$ distribution, independent of $B(t)$. Let $X(t) = \xi(1 + B(t))$. Show that $X(t)$ is Markov but not strongly Markov (w.r.t. the natural filtration).

**Solution:** First the interpretation: we toss a fair coin. If the result is "tails" (denoted as $\xi = 0$), then $X(t)$ is constant 0. If the result is "heads" (denoted as $\xi = 1$), then $X(t)$ is a Brownian motion starting from 1.

a.) This $X(t)$ is clearly not strongly Markov: if $\tau := \inf\{t \geq 1 \,|\, X(t) = 0\}$ is the first hitting time of 0 after time 1, then $X(\tau) = 0$ deterministically, so $\sigma(X(\tau))$ is the trivial (indiscrete) $\sigma$-algebra, containing no information, so

$$\mathbf{E}\big(F(X(\tau + u)) \;\big|\; \sigma(X(\tau))\big) = \mathbf{E}\left(F(X(\tau + u))\right)$$

is a constant for any bounded and measurable $F : \mathbb{R} \to \mathbb{R}$. As an example, let $F : \mathbb{R} \to \mathbb{R}$ be the indicator function of 0. Then, for any $u > 0$

$$\mathbf{E}\big(F(X(\tau + u)) \;\big|\; \sigma(X(\tau))\big) = \mathbf{E}\left(F(X(\tau + u))\right) = \mathbf{P}\left(X(\tau + u) = 0\right) = \frac{1}{2}$$

(since $\mathbf{P}\big(X(\tau + u) = 0 \;\big|\; \xi = 0\big) = 1$ and $\mathbf{P}\big(X(\tau + u) = 0 \;\big|\; \xi = 1\big) = 0$). On the other hand $\mathcal{F}_\tau$ is not trivial: it definitely contains the events $\{\xi = 0\}$ and $\{\xi = 1\}$. (If you see the trajectory up to $\tau$, you can tell the result of the coin toss.) So $\mathbf{E}\big(F(X(\tau+u)) \mid \mathcal{F}_\tau\big)$ is *not* constant (in particular it is 1 on $\{\xi=0\}$ and 0 on $\{\xi=1\}$).

In summary: for this particular $\tau$ and $F$, and for any $u>0$

$$\mathbf{E}\big(F(X(\tau+u)) \mid \mathcal{F}_\tau\big) \neq \mathbf{E}\big(F(X(\tau+u)) \mid \sigma(X(\tau))\big),$$

so the process is not strongly Markov.

b.) The surprising part is that $X(t)$ is Markov. This is because, for every fixed *deterministic* $t\in\mathbb{R}$, $\mathbf{P}\left(1+B(t)=0\right)=0$. This way if we see that $X(t)=0$, then we know that $\xi=0$ (or a zero probability event has occured). So

$$X(t+u)=\begin{cases}0 & \text{almost surely on } \{X(t)=0\}\\ 1+B(t+u) & \text{surely on } \{X(t)\neq 0\}.\end{cases}$$

This means

$$\begin{aligned}\mathbf{E}\big(F(X(t+u)) \mid X(t)\big) &= \begin{cases}F(0) & \text{a.s. on } \{X(t)=0\}\\ \mathbf{E}\big(F(1+B(t+u)) \mid B(t)\big) & \text{a.s. on } \{X(t)\neq 0\}.\end{cases}\\ &= \begin{cases}F(0) & \text{a.s. on } \{\xi=0\}\\ \mathbf{E}\big(F(1+B(t+u)) \mid B(t)\big) & \text{a.s. on } \{\xi=1\}.\end{cases}\end{aligned}$$

We need to see that $\mathbf{E}\big(F(X(t+u)) \mid \mathcal{F}_t\big)$ is the same. This can be seen by using that

- $\mathcal{F}_t=\sigma(\mathcal{F}_t^B,\xi)$,
- $X(t+u)=\xi(1+B(t+u))$ where $\xi$ is $\mathcal{F}_t$-measurable,
- $B(t+u)$ is independent of $\xi$,
- and $B(t)$ is Markov.

**2.10** a.) Show that if $X(t)$ is a submartingale, $\psi:\mathbb{R}\to\mathbb{R}$ is convex and *increasing* such that $\mathbf{E}\left(|\psi(X(t)|\right)<\infty$ for every $t$, then $Y(t):=\psi(X(t))$ is also a submartingale.

b.) Give an example of a submartingale $X(t)$ such that $Y(t):=(X(t))^2$ is not a submartingale.

**Solution:**

a.) i.) Adaptedness is understood w.r.t the natural filtration, so it is automatic.

ii.) Integrability is assumed explicitly as $\mathbf{E}\left(|Y(t)|\right)=\mathbf{E}\left(|\psi(X(t))|\right)<\infty$.

iii.) The essence is the submartingale property: for $t, u \geq 0$

$$\mathbf{E}\big(Y(t+u) \mid \mathcal{F}_t\big) = \mathbf{E}\big(\psi(X(t)) \mid \mathcal{F}_t\big) \overset{(1)}{\geq} \psi\left(\mathbf{E}\big(X(t) \mid \mathcal{F}_t\big)\right) \overset{(2)}{\geq} \psi(X(t)) = Y(t).$$

In step (1) we used Jensen's inequality. In step (2) we used that $\mathbf{E}\big(X(t) \mid \mathcal{F}_t\big) \geq X(t)$ by the submartingale property of $X(t)$ *and the monotonicity of* $\psi$.

b.) Note that $t \in [0, \infty)$. The increasing deterministic function $X(t) = -\frac{1}{t}$ does the job, since it's a submartingle, but $Y(t) := (X(t))^2 = \frac{1}{t^2}$ is (strictly) decreasing, so it's a supermartingale (and not a submartingale).

**2.11** Let $B(t)$ be a standard Brownian motion and let $X(t) = B(t) - \frac{t}{2}$: a kind of "Brownian motion with drift to the left". Let $-a < 0 < b$, let $\tau_{left} = \inf\{t \in \mathbb{R}^+ \mid X(t) = -a\}$ and $\tau_{right} = \inf\{t \in \mathbb{R}^+ \mid X(t) = b\}$ be the first hitting times for $-a$ and $b$, and let $\tau = \inf\{\tau_{left}, \tau_{right}\}$. Let $p_{left} = p_{left}(a, b) = \mathbf{P}\left(\tau_{left} < \tau_{right}\right)$ be the probability that $-a$ is reached sooner than $b$, and $p_{right} = p_{right}(a, b) = \mathbf{P}\left(\tau_{right} < \tau_{left}\right)$ be the probability that $b$ is reached sooner than $-a$.

a.) Show that $p_{left} + p_{right} = 1$, which means exactly that either $-a$ or $b$ is almost surely reached. (This is the same as saying that $\tau < \infty$ almost surely.)

b.) Find a number $q > 0$ such that $M(t) := q^{X(t)}$ is a martingale.

c.) Apply the optional stopping theorem to $M(t)$ and $\tau$ to find $p_{left}$ and $p_{right}$.

d.) Find the probability that $X(t)$ ever reaches $+1$. *(Hint: set $b = 1$, and look at $\lim p_{right}(a, b)$ as $a \to \infty$.)*

**Solution:**

a.) Clearly $\tau \leq \tau_{left}$, and I claim that $\tau_{left} < \infty$ almost surely. Indeed, for big $t$ the particle is very likely to be left of $-a$, because the expected position is $-\frac{t}{2}$, while the fluctuation around that is only around $\sqrt{t}$. With a rigorous calculation:

$$\mathbf{P}\left(X(t) > -a\right) = \mathbf{P}\left(\mathcal{N}\left(-\frac{t}{2}, t\right) > -a\right) = 1 - \Phi\left(\frac{-a - \left(-\frac{t}{2}\right)}{\sqrt{t}}\right) =$$

$$= 1 - \Phi\left(\frac{\sqrt{t}}{2} - \frac{a}{\sqrt{t}}\right) \to 0,$$

so

$$\mathbf{P}\left(X(t) > -a \text{ for every } t > 0\right) = 0.$$

b.) We see from Exercise 2.6 (with $\theta = 1$) that $q = e$ does the job: $M(t) := e^{X(t)} = \exp\{B(t) - \frac{t}{2}\}$ is a martingale.

c.) For $t \le \tau$ we have $B(t) \le b$, so $B(t \wedge \tau) \le b$, implying that $0 < M(t \wedge \tau) \le e^b$, so the stopped martingale is bounded. In part a.) we have seen that $\tau < \infty$ almost surely, so the optional stoping theorem can be applied, and it gives that $\mathbf{E}\left(M(\tau)\right) = M(0) = 1$. But $X(\tau) = -a$ on $\{\tau_{left} < \tau_{right}\}$ and $X(\tau) = b$ on $\{\tau_{right} < \tau_{left}\}$, so $1 = \mathbf{E}\left(M(\tau)\right) = \mathbf{E}\left(e^{X(\tau)}\right) = p_{left}e^{-a} + p_{right}e^b$. Together with part a.) we have the system of equations

$$\begin{cases} p_{left} + \quad p_{right} = 1 \\ e^{-a} p_{left} + e^b \quad p_{right} = 1 \end{cases}.$$

The unique solution is

$$p_{left} = \frac{e^b - 1}{e^b - e^{-a}}$$
$$p_{right} = \frac{1 - e^{-a}}{e^b - e^{-a}}.$$

d.) A fixed $b := 1 > 0$ is reached if and only if $b$ is reached sooner than $-a$ for some $a > 0$. So

$$\mathbf{P}\left(\{b = 1 \text{ is reached}\}\right) = \lim_{a \to \infty} p_{right}(a, b = 1) = \lim_{a \to \infty} \frac{1 - e^{-a}}{e - e^{-a}} = \frac{1}{e}.$$