Stochastic Differential Equations
Problem Set 3
The Itô integral, Itô's formula

3.1 Let $s \mapsto v(s)$ be a smooth deterministic function with $\sup_{0\le s\le T} |\, v'(s)\, | \le C$. Prove directly from the definition of the Itô integral that

$$\int_0^t v(s)dB(s) = v(t)B(t) - \int_0^t v'(s)B(s)ds.$$

Hint: Write

$$v(s_{i+1})B(s_{i+1}) - v(s_i)B(s_i) = v(s_i)(B(s_{i+1}) - B(s_i)) + B(s_{i+1})(v(s_{i+1})) - v(s_i)).$$

3.2 Prove directly form the definition of the Itô integral that

$$\int_0^t B(s)dB(s) = \frac{1}{2}B(t)^2 - \frac{t}{2},$$

$$\int_0^t B(s)^2 dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)ds.$$

3.3 Suppose $v, w \in \mathcal{V}_T$ and $C, D \in \mathbb{R}$ are such that

$$\int_0^T v(s)dB(s) + C = \int_0^T w(s)dB(s) + D.$$

Show that $C = D$ and $v = w$ (s, ω)-almost surely.

3.4 (a) For which values of $\alpha \in \mathbb{R}$ is the process

$$Y_\alpha(t) := \int_0^t (t-s)^{-\alpha} dB(s)$$

well defined as an Itô integral?.

(b) Compute the covariances $\mathbf{E}\,(Y_\alpha(s)Y_\alpha(t))$.

Remark: The process $t \mapsto Y_\alpha(t)$ is called *fractional Brownian motion.*

3.5 Use Itô's formula to write the following processes $t \mapsto X(t)$ in the standard form

$$X(t) = X(0) + \int_0^t u(s)ds + \int_0^t v(s)dB(s).$$

Identify the processes $s \mapsto u(s)$ and $s \mapsto v(s)$ under the integrals. Notation: $B(t)$ denotes standard 1-dimensional Brownian motion, $(B_1(t), \ldots, B_n(t))$ denotes standard n-dimensional Brownian motion (that is: n independent standard 1-dimensional Brownian motions).

(a) $X(t) = B(t)^2$

(b) $X(t) = 2 + t + e^{B(t)}$

(c) $X(t) = B_1(t)^2 + B_2(t)^2$

(d) $X(t) = (t, B(t))$

(e) $X(t) = (B_1(t) + B_2(t) + B_3(t), B_2(t)^2 - B_1(t)B_3(t))$

3.6 Use Itô's formula to prove that

$$\int_0^t B(s)^2 dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)ds.$$

3.7 Suppose $\theta(t) = (\theta_1(t), \ldots, \theta_n(t)) \in \mathbb{R}^n$ with $t \mapsto \theta_j(t)$, $j = 1, \ldots, n$, progressively measurable and a.s. bounded in any compact interval $[0, T]$. Define

$$Z(t) := \exp\left\{ \int_0^t \theta(s)dB(s) - \frac{1}{2}\int_0^t |\,\theta(s)\,|^2\, ds \right\},$$

where $t \mapsto B(t)$ is standard Brownian motion in $\mathbb{R}^n$ and $|\,\theta\,|^2 = \theta_1^2 + \cdots + \theta_n^2$.

(a) Use Itô's formula to prove that

$$dZ(t) = Z(t)\theta(t)dB(t).$$

(b) Deduce that $t \mapsto Z(t)$ is a martingale.

3.8 Let $t \mapsto B(t)$ be a standard 1-dimensional Brownian motion with $B(0) = 0$, and

$$\beta_k(t) := \mathbf{E}\left(B(t)^k\right).$$

Use Itô's formula to prove that

$$\beta_{k+2}(t) = \frac{1}{2}(k+2)(k+1)\int_0^t \beta_k(s)ds.$$

Compute explicitly $\beta_k(t)$ for $k = 0, 1, 2, \ldots, 6$.

3.9 Let $t \mapsto B(t)$ be a standard one-dimensional Brownian motion and $r, \alpha \in \mathbb{R}$ constants. Define

$$X(t) := \exp\{\alpha B(t) + rt\}.$$

Prove that

$$dX(t) = (r + \frac{\alpha^2}{2})X(t)dt + \alpha X(t)dB(t).$$

3.10 Let $t \mapsto B(t) \in \mathbb{R}^m$ be standard m-dimensional Brownian motion, $t \mapsto v(t) \in \mathbb{R}^{n\times m}$ progressively measurable and a.s. bounded. Define

$$X(t) = \int_0^t v(s)dB(s) \in \mathbb{R}^n.$$

Prove that

$$M(t) := |\, X(t) \,|^2 - \int_0^t \operatorname{tr}\{v(s)v(s)^T\}ds$$

is a martingale.

3.11 Use Itô's formula to prove that the following processes are $(\mathcal{F}_t^B)$-martingales.

(a) $X(t) = e^{t/2}\cos B(t)$

(b) $X(t) = e^{t/2}\sin B(t)$

(c) $X(t) = (B(t) + t)\exp\{-B(t) - t/2\}$

3.12 Let $t \mapsto u(t)$ be progressively measurable and almost surely bounded. Define

$$X(t) := \int_0^t u(s)ds + B(t),$$

$$M(t) := \exp\left\{-\int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u(s)^2 ds\right\}$$

(Note that according to the statement of problem 7 the process $t \mapsto M(t)$ is a martingale.) Prove that the process

$$t \mapsto Y(t) := X(t)M(t)$$

is a $(\mathcal{F}_t^B)$-martingale.

3.13 In each of the cases below find a process $t \mapsto v(t)$ such that $v \in \mathcal{V}_T$ and the random variable X is written as

$$X = \mathbf{E}\,(X) + \int_0^T v(s)dB(s).$$

$$(a)\quad X = B(T), \qquad (b)\quad X = \int_0^T B(s)ds, \qquad (c)\quad X = B(T)^2,$$

$$(d)\quad B(T)^3, \qquad (e)\quad e^{B(T)}, \qquad (f)\quad \sin B(T).$$

3.14 Let $x \geq 0$ and define the process

$$X(t) := (x^{1/3} + \frac{1}{3}B(t))^3.$$

Show that

$$dX(t) = \frac{1}{3}\mathrm{sgn}(X(t)) \,|\, X(t) \,|^{1/3} dt + |\, X(t) \,|^{2/3} dB(t), \qquad X(0) = x.$$

3.15 Let $0 = t_0 < t_1 < \cdots < t_n = 1$ and let $\phi_0, \phi_1, \ldots, \phi_{n-1}$ be random variables with finite variance. Then the random function $\psi : [0,1] \to \mathbb{R}$ defined as

$$\psi(t) = \sum_{i=1}^{n} \phi_{i-1} \mathbb{1}_{[t_{i-1}, t_i)}(t) \tag{1}$$

is called *simple*. The stochastic integral of this ψ is defined as

$$\int_0^1 \psi(t)\mathrm{d}B(t) := \sum_{i=1}^{n} \phi_{i-1}(B(t_i) - B(t_{i-1})).$$

Now let's try to calculate $\int_0^1 B(t)\mathrm{d}B(t)$ by approximating $B(t)$ with simple functions $\psi_1, \psi_2, \ldots$: this means that we calcluate the L^2 limit

$$I := \lim_{n\to\infty} \int_0^1 \psi_n(t)\mathrm{d}B(t),$$

where $\psi_n(t) \to B(t)$. In this exercise, let's do this with ψ_n defined as (1) with $t_i = \frac{i}{n}$ and $\phi_{i-1} = B\left(\frac{t_{i-1}+t_i}{2}\right)$.

a.) Calculate this random variable I.

(Hint: Let $X_k = B\left(\frac{k}{2n}\right) - B\left(\frac{k-1}{2n}\right)$. *Then*

$$\sum_{i=1}^{n} \phi_{i-1}(B(t_i) - B(t_{i-1})) = \sum_{i=1}^{n} B(t_{i-1})(B(t_i) - B(t_{i-1})) + \sum_{i=1}^{n} X_{2i-1}(X_{2i-1} + X_{2i}).$$

The first sum should be familiar from the lecture. For the second, calculate the expectation and the variance to get the L^2 *limit.)*

b.) Why is this not equal to the Itô integral $\int_0^1 B(t)\mathrm{d}B(t)$?

Solution:

a.) From the text of the exercise: we are interested in $I = \lim_{n\to\infty} I_n$ where

$$I_n = \sum_{i=1}^{n} \phi_{i-1}(B(t_i) - B(t_{i-1})).$$

(On the right hand side the ϕ_i and t_i depend on n, although this is not explicitly indicated.) According to the hint, we write this as $I_n = I_n^{(1)} + I_n^{(2)}$ where

$$I_n^{(1)} = \sum_{i=1}^{n} B(t_{i-1})(B(t_i) - B(t_{i-1})),$$

$$I_n^{(2)} = \sum_{i=1}^{n} X_{2i-1}(X_{2i-1} + X_{2i}).$$

Now $I_n^{(1)}$ is the approximating sum for the Itô integral $\int_0^1 B(t)\mathrm{d}B(t)$, so

$$I^{(1)} := \lim_{n\to\infty} I_n^{(1)} = \int_0^1 B(t)\mathrm{d}B(t) = \frac{1}{2}B^2(1) - 1 \quad \text{(in } L^2\text{)}.$$

We are left to calculate the L^2 limit $I^{(2)} := \lim_{n\to\infty} I_n^{(2)}$. The X_k are independent and $X_k \sim \mathcal{N}(0, \frac{1}{2n})$, so

$$\mathbf{E}\left(I_n^{(2)}\right) = \sum_{i=1}^{n} \mathbf{E}\left(X_{2i-1}^2 + X_{2i-1}X_{2i}\right) = \sum_{i=1}^{n}\left(\frac{1}{2n} + 0\right) = \frac{1}{2}.$$

To calculate the variance, observe that the terms $Y_i := X_{2i-1}(X_{2i-1} + X_{2i})$ are also independent for the different values of i – e.g. Y_1 depends on X_1, X_2 while Y_2 depends on X_3, X_4. They are also identically distributed, so

$$\mathbf{Var}\left(I_n^{(2)}\right) = n\mathbf{Var}\left(Y_1\right) = n\mathbf{Var}\left(X_1(X_1 + X_2)\right).$$

Write $X_1 = \frac{1}{\sqrt{2n}}\xi$, $X_2 = \frac{1}{\sqrt{2n}}\eta$ where $\xi, \eta \sim \mathcal{N}(0,1)$ are independent. Then

$$\begin{aligned}\mathbf{Var}\left(I_n^{(2)}\right) &= n\mathbf{Var}\left(\frac{1}{\sqrt{2n}}\xi\left(\frac{1}{\sqrt{2n}}\xi + \frac{1}{\sqrt{2n}}\eta\right)\right) \\ &= n\mathbf{Var}\left(\frac{\xi(\xi+\eta)}{2n}\right) = \frac{n}{4n^2}\mathbf{Var}\left(\xi(\xi+\eta)\right).\end{aligned}$$

The exact value is not important, we only need that $\mathbf{Var}\left(I_n^{(2)}\right) \to 0$ as $n \to \infty$. With the expectation being constant $\frac{1}{2}$, this proves that $I_n^{(2)} \to I^{(2)} = \frac{1}{2}$ in L^2. So

$$I = I^{(1)} + I^{(2)} = \frac{1}{2}B^2(1) - \frac{1}{2}.$$

b.) In the definition of the Itô integral, the simple functions used must be *progressively measurable*, which implies that they must be *adapted* to the filtration $\mathcal{F}_t$. This $\mathcal{F}_t$ is usually the natural filtration of $B(t)$, but even if not, it is definitely required that $B(t)$ be a Brownian motion *with respect to this filtartion*, meaning e.g. that $B(t+h) - B(t)$ is independent of $\mathcal{F}_t$ for any $t, h > 0$.

Now, the simple functions in this exercise are definitely *not* like that:

$$\psi(t_{i-1}) = \phi_{i-1} = B\left(\frac{t_{i-1} + t_i}{2}\right)$$

contains information about the future of $B(t)$ because $\frac{t_{i-1}+t_i}{2} > t_{i-1}$.

3.16 Let $B(t)$ be a standard Brownian motion and let $\mathcal{F}_t$ be its natural filtration. Let the random variable X be $X = \mathbb{1}_{\{B(1)\geq 0\}}$. Of course, $X \in \mathcal{F}_1$.

a.) For $0 \leq t \leq 1$ let $M(t) = \mathbf{E}\big(X \bigm| \mathcal{F}_t\big)$. Calculate $M(t)$ explicitly for $0 \leq t < 1$. *(Hint: since X is a function of $B(1)$ and B is Markov, $M(t)$ is a function of t and $B(t)$: write it as $M(t) = f(t, B(t))$ with some function $f(t,x)$.)*

b.) Check directly that $M(t) \to X$ almost surely as $t \nearrow 1$.

c.) Use the Itô formula to check directly that $M(t)$ is a martingale.

d.) Notice that you just found the Itô representation of $M(t)$ as well as the Itô representation of: X: write it as $X = \mathbf{E}(X) + \int_0^1 v(t)\mathrm{d}B(t)$ with an appropriate process v.

e.) Calculate directly the integral $V := \int_0^1 \mathbf{E}\left(v(t)^2\right) \mathrm{d}t$, and check that the Itô isometry holds.

Solution:

a.) Let $0 \leq t < 1$ and $x \in \mathbb{R}$ fixed. Under the condition $B(t) = x$, the conditional expectation of X is

$$\mathbf{P}\big(B(1) \geq 0 \bigm| B(t) = x\big) = \mathbf{P}\left(B(1) - B(t) \geq -x\right) = 1 - \Phi\left(\frac{-x}{\sqrt{1-t}}\right)$$
$$= \Phi\left(\frac{x}{\sqrt{1-t}}\right)$$

(where Φ is the standard Gaussion distribution function) because $B(1)-B(t) \sim \mathcal{N}(1-t)$ is independent of $B(t)$. So

$$M(t) = \mathbf{E}\big(X \mid \mathcal{F}_t\big) = \Phi\left(\frac{B(t)}{\sqrt{1-t}}\right).$$

b.) $t \mapsto B(t)$ is almost surely continuous in t, so

- If $B(1) > 0$ then $\frac{B(t)}{\sqrt{1-t}} \to \frac{B(1)}{+0} = \infty$ (almost surely) as $t \nearrow 1$. This implies that $M(t) = \Phi\left(\frac{B(t)}{\sqrt{1-t}}\right) \to 1$.
- If $B(1) < 0$ then $\frac{B(t)}{\sqrt{1-t}} \to \frac{B(1)}{+0} = -\infty$ (almost surely) as $t \nearrow 1$. This implies that $M(t) = \Phi\left(\frac{B(t)}{\sqrt{1-t}}\right) \to 0$.

In both cases, the limit coincides with $X = \mathbb{1}_{\{B(1)\geq 0\}}$. If $B(1) = 0$ then $\frac{B(t)}{\sqrt{1-t}}$ typically does not converge, but this happens with probability 0, so $M(t) \to X$ almost surely as $t \nearrow 1$.

c.) $M(t) = f(t, B(t))$ with $f(t,x) = \Phi\left(\frac{x}{\sqrt{1-t}}\right)$, so we need

$$\begin{aligned}
\partial_t f(t,x) &= \varphi\left(\frac{x}{\sqrt{1-t}}\right)\frac{x}{2\sqrt{1-t}^3} \\
\partial_x f(t,x) &= \varphi\left(\frac{x}{\sqrt{1-t}}\right)\frac{1}{\sqrt{1-t}} \\
\partial^2_{x^2} f(t,x) &= -\frac{x}{\sqrt{1-t}}\varphi\left(\frac{x}{\sqrt{1-t}}\right)\left(\frac{1}{\sqrt{1-t}}\right)^2 = -\varphi\left(\frac{x}{\sqrt{1-t}}\right)\frac{x}{\sqrt{1-t}^3}.
\end{aligned}$$

Here φ denotes the standard Gaussian density function and we have used that $\varphi'(y) = -y\varphi(y)$. Writing these back to Itô's formula, we happily see that $\partial_t f(t,x) + \frac{1}{2}\partial^2_{x^2} f(t,x) = 0$, so the coefficient of $\mathrm{d}t$ vanishes and

$$\mathrm{d}M(t) = 0\mathrm{d}t + \frac{1}{\sqrt{1-t}}\varphi\left(\frac{B(t)}{\sqrt{1-t}}\right)\mathrm{d}B(t),$$

which is the same as

$$M(t) = M(0) + \int_0^t \frac{1}{\sqrt{1-s}}\varphi\left(\frac{B(s)}{\sqrt{1-s}}\right)\mathrm{d}B(s).$$

Of course, $M(0) = \mathbf{E}\left(X\right) = \mathbf{P}\left(()\, B(1) \geq 0\right) = \frac{1}{2}$.

So indeed, we found $M(t)$ to be an Itô integral, so it's a martingale.

d.) With $v(t) := \frac{1}{\sqrt{1-t}}\varphi\left(\frac{B(t)}{\sqrt{1-t}}\right)$, we found that

$$M(t) = \frac{1}{2} + \int_0^t v(s)\mathrm{d}B(s),$$
$$X = \frac{1}{2} + \int_0^1 v(t)\mathrm{d}B(t).$$

e.) $B(t)$ has density $f_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$, so

$$\begin{aligned}\mathbf{E}\left(v(t)^2\right) &= \mathbf{E}\left(\left[\frac{1}{\sqrt{1-t}}\varphi\left(\frac{B(t)}{\sqrt{1-t}}\right)\right]^2\right) = \int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{1-t}}\varphi\left(\frac{x}{\sqrt{1-t}}\right)\right]^2 f_t(x)\mathrm{d}x \\ &= \frac{1}{1-t}\frac{1}{\sqrt{t}}\frac{1}{\sqrt{2\pi}^3}\int_{-\infty}^{\infty} e^{-2\cdot\frac{1}{2}\left(\frac{x}{\sqrt{1-t}}\right)^2-\frac{x^2}{2t}}\mathrm{d}x = \frac{1}{1-t}\frac{1}{\sqrt{t}}\frac{1}{\sqrt{2\pi}^3}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{1+t}{t(1-t)}x^2}\mathrm{d}x \\ &= \frac{1}{1-t}\frac{1}{\sqrt{t}}\frac{1}{\sqrt{2\pi}^3}\sqrt{\frac{2\pi t(1-t)}{1+t}} = \frac{1}{2\pi}\frac{1}{\sqrt{1-t^2}}.\end{aligned}$$

(We have used that $\int_{-\infty}^{\infty} e^{-ax^2}\mathrm{d}x = \sqrt{\frac{\pi}{a}}$ for $a > 0$.) So

$$\int_0^1 \mathbf{E}\left(v(t)^2\right)\mathrm{d}t = \frac{1}{2\pi}\int_0^1 \frac{1}{\sqrt{1-t^2}}\mathrm{d}t = \frac{1}{2\pi}[\arcsin t]_0^1 = \frac{1}{2\pi}\frac{\pi}{2} = \frac{1}{4}.$$

On the other hand $X \sim B\left(\frac{1}{2}\right)$, so $\mathbf{Var}\left(X\right) = \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{4}$, so the Itô isometry holds:

$$\int_0^1 \mathbf{E}\left(v(t)^2\right)\mathrm{d}t = \mathbf{Var}\left(X\right).$$

3.17 Let $B(t)$ be a standard Brownian motion and let $\mathcal{F}_t$ be its natural filtration. Let T be the time spent by $B(t)$ on the positive half-line up to $t = 1$:

$$T := \mathit{Leb}(\{s \in [0,1] \mid B(s) \geq 0\}).$$

We will find the Itô representation of T – that is, the process $u(t)$ for which

$$T = \mathbf{E}\left(T\right) + \int_0^1 u(t)\mathrm{d}B(t).$$

a.) For every $0 \leq s \leq 1$ let $X_s = \mathbb{1}_{\{B(s)\geq 0\}}$. Then $T = \int_0^1 X_s \mathrm{d}s$. As in the previous exercise, find the martingale $t \mapsto M_s(t) := \mathbf{E}\big(X_s \bigm| \mathcal{F}_t\big)$, $0 \leq t \leq 1$. *(Warning: the formula will be different for $t < s$ and $t \geq s$.)*

b.) As in the previous exercise, find the Itô representation

$$X_s = \mathbf{E}(X_s) + \int_0^1 v_s(t)\mathrm{d}B(t).$$

(Warning: you will encounter the integral $\int_s^1 \frac{1}{\sqrt{t-s}}\varphi\left(\frac{x}{\sqrt{t-s}}\right)$ *where* φ *is the standard normal density function. If you want to use substitution, be careful: the sign of* x *matters.)*

c.) Use that $T = \int_0^1 X_s \mathrm{d}s$ to find the Itô representation of T.

d.) ** An alternative way is to use the martingale

$$N(t) := \mathbf{E}\big(T \mid \mathcal{F}_t\big) = \int_0^1 \mathbf{E}\big(X_s \mid \mathcal{F}_t\big).$$

Calculate $N(t)$, use Itô's formula to check that it is really a martingale and find its Itô representation, which is also the Itô representation of T. Check that you got the same as with the previous method.