Stochastic Processes, Problem Set 1: Solutions.

1.1

$$\frac{d}{dx}\left\{\left(\frac{1}{x}-\frac{1}{x^3}\right)\varphi(x)\right\} = \cdots = \left(-1+\frac{3}{x^4}\right)\varphi(x)$$

$$\vee$$

$$\frac{d}{dx}\left\{1-\Phi(x)\right\} = -\varphi(x)$$

$$\vee$$

$$\frac{d}{dx}\left\{\frac{1}{x}\varphi(x)\right\} = \cdots = \left(-1-\frac{1}{x^2}\right)\varphi(x)$$

Since

$$\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{x^2}\right)\varphi(x) = \lim_{x\to\infty}\left(1-\Phi(x)\right) = \lim_{x\to\infty}\frac{1}{x}\varphi(x) = 0$$

the claimed inequalities follow $\square$

1.2

a
$$\mathbb{E}\left(B^{(n)}(t)\right) = \sum_{j=1}^{\lfloor nt\rfloor}\mathbb{E}\left(X_j^{(n)}\right) = 0$$

$$\mathrm{Cov}\left(B^{(n)}(t), B^{(n)}(s)\right) = \mathbb{E}\left(B^{(n)}(t)\,B^{(n)}(s)\right) =$$

$$= \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} E\left(X_j^{(n)} X_i^{(n)}\right) = \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} \frac{1}{n} \delta_{ij}$$

$$= \frac{1}{n} \cdot \min\left(\lfloor nt \rfloor, \lfloor ns \rfloor\right) \to \min(s, t)$$

as $n \to \infty$.

b The collection of random variables $\{B^{(n)}(t) : t \in [0,1]\}$ is jointly Gaussian with expectations and covariances given in (a)

c $\delta_n = \max_{1 \le j \le n} |X_j^{(n)}|$

subadditivity of probabilities

$$\mathbb{P}\left(\max_{1 \le j \le n} |X_j^{(n)}| \geq \varepsilon\right) \leq$$

$X_j^{(n)} \sim \frac{1}{\sqrt{n}} \xi$ ← $\mathcal{N}(0,1)$

$$\sum_{j=1}^{n} \mathbb{P}\left(|X_j^{(n)}| \geq \varepsilon\right) =$$

problem 1.1

$$= n \, \mathbb{P}\left(\frac{1}{\sqrt{n}} \xi \geq \varepsilon\right) = n\left(1 - \Phi(\sqrt{n}\varepsilon)\right) \leq \ldots$$

$$n\left(1-\Phi(\sqrt{n}\,\varepsilon)\right) \underset{\text{by problem 1.1}}{\leq} \frac{\sqrt{n}}{\sqrt{2\pi}\,\varepsilon} e^{-\varepsilon^2 n/2} \longrightarrow 0$$

as $n \to \infty$

$\square$

1.3

a $$\mathbb{E}\left(Z^{(n)}(t)\right) = \sum_{j=1}^{\lfloor nt \rfloor} \mathbb{E}\left(Y_j^{(n)} - \frac{1}{n}\right) = 0$$

$$\mathrm{Cov}\left(Z^{(n)}(t), Z^{(n)}(s)\right) = \sum_{j=1}^{\lfloor nt \rfloor} \sum_{i=1}^{\lfloor ns \rfloor} \mathrm{Cov}\left(Y_j^{(n)}, Y_i^{(n)}\right) =$$

$$= \sum_{j=1}^{\lfloor nt \rfloor} \sum_{i=1}^{\lfloor ns \rfloor} \frac{1}{n} \delta_{ij} = \frac{1}{n} \min\left(\lfloor ns \rfloor, \lfloor nt \rfloor\right) \xrightarrow[\text{as } n\to\infty]{} \min(s,t)$$

b Let $0 = t_0 < t_1 < t_2 < \dots < t_n = 1$

Then

$\left\{ Z^{(n)}(t_j) - Z^{(n)}(t_{j-1}) \;:\; j = 1, \dots, n \right\}$ are independent

and the distribution of $Z^{(n)}(t) - Z^{(n)}(s)$ is

$\mathrm{POI}\left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)$ centred (expectation subtracted) c

$$\delta_n = \max_{1 \le j \le n} Y_j^{(n)} - \frac{1}{n}$$

for any $\varepsilon < 1$: let $n > \frac{1}{1-\varepsilon}$

Then

$$\mathbb{P}(\delta_n > \varepsilon) = \mathbb{P}\left(\max_{1 \le j \le n} Y_j^{(n)} \ge 1\right) =$$

$$1 - \mathbb{P}\left(Y_j^{(n)} = 0,\ j = 1, \dots, n\right) = 1 - \left(e^{-\frac{1}{n}}\right)^n$$

$$= 1 - e^{-1}$$

$\square$

1.4 $B^{(n)}(t)$ is "almost continuous"

while

$Z^{(n)}(t)$ has jumps of order 1

$B^{(n)}(t) \Longrightarrow B(t)$ while $Z^{(n)}(t) \Rightarrow \text{Poisson}(t)$

in some sense

in some sense

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$C = (c_{ij})_{i,j=1}^{n}$ $c_{ij} = Cov(Y_i, Y_j) = \mathbb{E}(Y_i Y_j)$

C is $\boxed{\text{positive definite}}$

$\exists!\ A = (a_{ij})_{i,j=1}^{n}$ $\boxed{\text{positive definite}}$ matrix, real

such that $C = A^2$. Denote $B := A^{-1}$

if $\underline{Y} = (Y_1, \dots Y_n)^T$ is jointly Gaussian

let $\underline{\underline{X}} = B\,\underline{Y}$ $\underline{X} = (X_1, \dots, X_n)^T$

As linear combinations of jointly Gaussians, $(X_i)_{i=1}^{n}$ <u>are</u> jointly Gaussian, too.

$$\mathbb{E}(X_i X_j) = \sum_{k,l} B_{ik} B_{jl}\ \mathbb{E}(Y_k Y_l) =$$

$$(BCB)_{ij} = \delta_{ij}$$

So the Gaussian random variables $(X_i)_{i=1}^{n}$ are uncorrelated and thus independent

1.6 $X(t), Y(t), Z(t)$ are Gaussian processes since they are <u>linear</u> expressions of $B(t)$.

$$\mathbb{E}(X(t)) = \mathbb{E}(Y(t)) - \mathbb{E}(Z(t)) = 0 \checkmark$$

$$\mathbb{E}(X(t)X(s)) = a^{-1}\,\mathbb{E}(B(at)B(as)) =$$

$$a^{-1}\min(as, at) = \min(s, t) \checkmark$$

$$\mathbb{E}(Y(t)Y(s)) = t\cdot s\,\mathbb{E}\left(B(1/t)\,B(1/s)\right)$$

$$= t s \cdot \min\left(\frac{1}{t}, \frac{1}{s}\right) = \min(t, s)$$

$$\mathbb{E}(Z(t)Z(s)) = \mathbb{E}\left(\left(B(T) - B(T-t)\right)\left(B(T) - B(T-s)\right)\right)$$

$$= \mathbb{E}(B(T)B(T)) - \mathbb{E}(B(T)B(T-s)) -$$

$$- \mathbb{E}(B(T-t)B(T)) + \mathbb{E}(B(T-t)B(T-s)) =$$

$$T - (T-s) - (T-t) + \min(T-t, T-s) =$$

$$= \min(t, s) \qquad \square$$

Continuity:

$t \mapsto X(t)$ is continuous on $t \in [0, \infty)$

$t \mapsto Y(t)$ — " — on $t \in (0, \infty)$

$t \mapsto Z(t)$ — " — on $t \in [0, T]$

$\boxed{\text{straightforward}}$

Continuity of $t \mapsto Y(t)$ at $t = 0$ needs proof.

Since $t \mapsto Y(t)$ is continuous on $(0, \infty)$

$$\left\{ \lim_{t \downarrow 0} Y(t) = 0 \right\} =$$

$$\bigcap_{m > 0} \bigcup_{n > 0} \bigcap_{q \in (0, \frac{1}{n}) \cap \mathbb{Q}} \left\{ |Y(q)| < \frac{1}{n} \right\}$$

Since $(Y(t) : t > 0) \sim (B(t) : t > 0)$

it follows that

$$\mathbb{P}\left(\lim_{t \downarrow 0} Y(t) = 0\right) = \mathbb{P}\left(\lim_{t \downarrow 0} B(t) = 0\right) = \frac{1}{0}$$

(17)

17 straightforward: $Z(t)$ is Gaussian

$$E\big(Z(t)Z(s)\big) = \sum_{ij} a_i a_j \underbrace{E\big(B_i(t)B_j(s)\big)}_{=\delta_{ij}\min(t,s)} =$$

$$= \sum_{i=1}^{n} a_i^2 \min(t,s)$$

So, $t \mapsto Z(t)$ is BM with

$$\sigma^2 = \sum_{i=1}^{n} a_i^2 \quad \square$$

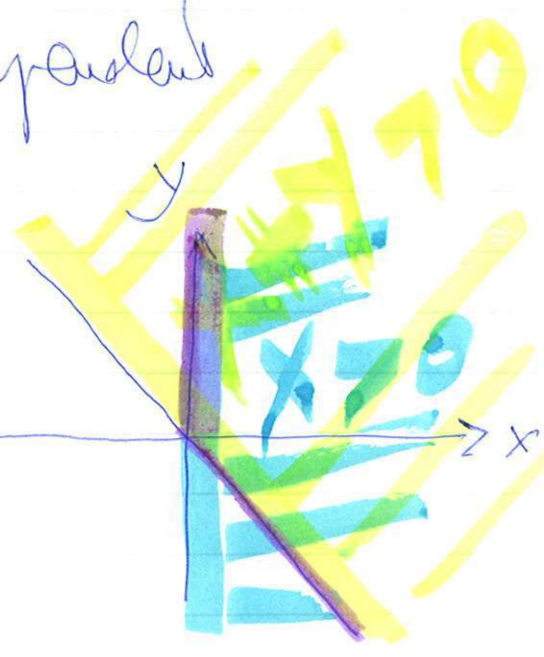
18 $P\big((B(2)-B(1)) + B(1) \geq 0 \mid B(1) \geq 0\big) =$

$$\mathbb{P}\big(X+Y \geq 0 \mid X \geq 0\big)$$

where X, Y are independent

$$\sim \mathcal{N}(0,1)$$

$$= \frac{3}{4} \text{ see picture}$$



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$$\mathbb{P}\left(\inf_{0\le t\le 1} B(t) = 0\right) =$$

$$\mathbb{P}\left(\bigcap_n \left\{\inf_{0\le t\le 1} B(t) > -\frac{1}{n}\right\}\right) =$$

$$\lim_{n\to\infty} \mathbb{P}\left(\inf_{0\le t\le 1} B(t) \geq -\frac{1}{n}\right) =$$

$$\lim_{n\to\infty} \mathbb{P}\left(M(1) \le \frac{1}{n}\right) =$$

$$\lim_{n\to\infty} \left(2\,\Phi\left(\frac{1}{n}\right) - 1\right) = 0.$$

similarly : $\mathbb{P}\left(\sup_{0\le t\le 1} B(t) = 0\right) = 0$

$\{B(t)$ doesn't change sign in $[0,1]\} =$

$$\left\{\inf_{0\le t\le 1} B(t) = 0\right\} \cup \left\{\sup_{0\le t\le 1} B(t) = 0\right\}$$

Thus:

$\mathbb{P}(B(t)$ doesn't change sign in $[0,1]) = 0.$

Let $\varepsilon > 0$. By scaling

$$\mathbb{P}\left(B(t) \text{ doesn't change sign in } [0, \varepsilon] \right) = 0$$

$$\mathbb{P}\left(B(t) \text{ doesn't change sign on } [0, \varepsilon] \text{ for some } \varepsilon > 0 \right)$$

$$= \mathbb{P}\left(\bigcup_n \{ B(t) \text{ doesn't change sign in } [0, \tfrac{1}{n}] \right) =$$

$$= \lim_{n \to \infty} \mathbb{P}\left(B(t) \text{ doesn't change sign in } [0, \tfrac{1}{n}] \right) = 0.$$

so: $\mathbb{P}\left(\inf\{ t > 0 : B(t) = 0 \} = 0 \right) = 1.$