4.1 Straightforward applications of Itô's formula.

4.2
$$\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = \begin{pmatrix} \cos B(t) \\ \sin B(t) \end{pmatrix}$$

by applying Itô's formula one gets

$$d\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} dt + \begin{pmatrix} -V(t) \\ U(t) \end{pmatrix} dB(t).$$

4.4

a) $$X(t) = X(0)e^{-\gamma t} + a\int_0^t e^{-\gamma(t-s)}dB(s)$$

b) $$\mathbb{E}(X(t)) = x_0 e^{-\gamma t}$$

let $0 \leq s \leq t < \infty$:

$$Cov(X(t), X(s)) =$$

$$a^2 \mathbb{E}\left(\int_0^t e^{-\gamma(t-u)}dB(u) \cdot \int_0^t e^{-\gamma(s-v)}dB(v)\right) =$$

$$a^2 \int_0^s e^{-\gamma(t+s-2u)}du =$$

$$\frac{a^2}{2\gamma}\left(e^{-\gamma|t-s|} - e^{-\gamma(t+s)}\right)$$

(c) $$\mathbb{E}\left(Y^{(n)}_{k+1} - Y^{(n)}_{k} \,\middle|\, Y^{(n)}_{k} = y\right) =$$

$$-\frac{2}{n}\left(y - \frac{n}{2}\right)$$

$$\mathrm{Var}\left(Y^{(n)}_{k+1} - Y^{n}_{k} \,\middle|\, Y^{(n)}_{k} = y\right) =$$

$$1 - \frac{4}{n^2}\left(y - \frac{n}{2}\right)^2$$

May think abou [illegible] s:

$$Y^{(n)}_{k+1} - Y^{(n)}_{k} = -\frac{2}{n}\left(Y^{(n)}_{k} - \frac{n}{2}\right) + \xi^{(n)}_{k+1} +$$

$$+ \text{small error} \; \cancel{\xi^{(n)}_{k+1}}$$

random variable independent of the past

$$\mathbb{E}\left(\xi^{(n)}_{k+1}\right) = 0$$

$$\mathrm{Var}\left(\xi^{(n)}_{k+1}\right) = 1$$

Now, let $X^{(n)}(t) := \frac{1}{\sqrt{n}}\left(Y^{(n)}_{\lfloor nt \rfloor} - \frac{n}{2}\right)$

$$dt \simeq \frac{1}{N}$$

to get

$$dX^{(n)}(t) \approx -2X^{(n)}(t)\,dt + dB(t) \ldots$$

4.5

(a) $Af(x) = \beta f'(x) + \frac{\alpha^2}{2} x^2 f''(x)$

(b) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \qquad f = f(u,x)$

$$Af(u,x) = \frac{\partial f}{\partial u}(u,x) - \gamma x \frac{\partial f}{\partial x}(u,x) + \frac{\alpha^2}{2} \frac{\partial^2 f}{\partial x^2}(u,x)$$

(c) $f : \mathbb{R}^2 \to \mathbb{R}$

$$Af(x_1,x_2) = \frac{\partial f}{\partial x_1}(x_1,x_2) + x_2 \frac{\partial f}{\partial x_2}(x_1,x_2) + \frac{1}{2} e^{2x_1} \frac{\partial^2 f}{\partial x_2^2}(x_1,x_2)$$

(d) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f = f(x_1,x_2)$

$$Af(x_1,x_2) = \frac{\partial f}{\partial x_1}(x_1,x_2) + \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2}(x_1,x_2) + \frac{\alpha^2}{2} \frac{\partial^2 f}{\partial x_2^2}(x_1,x_2)$$

4.6

(a) $dX(t) = dt + \sqrt{2}\, dB(t)$

(b) $$dY(t) = \begin{pmatrix} dt \\ cX(t)\, dt + \alpha X(t)\, dB(t) \end{pmatrix}$$

4.8

$\boxed{x > 0}$

$$Af(x) = \frac{\delta-1}{2x} f'(x) + \frac{1}{2} f''(x)$$

$Af(x) = 0$ general sln:

$$f(x) = \begin{cases} C_1 + C_2\, x^{2-\delta} & \delta \neq 2 \\ C_1 + C_2 \ln x & \delta = 2 \end{cases}$$

b) $0 < r < x < R < \infty$

$$\mathbb{P}_x(\tau_r < \tau_R) = \begin{cases} \dfrac{x^{2-\delta} - R^{2-\delta}}{r^{2-\delta} - R^{2-\delta}} & \text{if } \delta \neq 2 \\[2ex] \dfrac{\ln x - \ln R}{\ln r - \ln R} & \text{if } \delta = 2 \end{cases}$$

c $\boxed{\delta > 2}$

$$\lim_{R\to\infty} P(\tau_r < \tau_R) = \left(\frac{r}{x}\right)^{\delta-2} < 1$$

$$\underbrace{\lim_{r\to 0} \lim_{R\to\infty}}_{\text{in this order !!!}} P(\tau_r < \tau_R) = 0$$

d $\boxed{\delta = 2}$

$$\lim_{R\to\infty} P_x(\tau_r < \tau_R) = 1$$

$$\lim_{r\to 0} P_x(\tau_r < \tau_R) = 0$$

e $\boxed{\delta < 2}$

$$\lim_{r\to 0} P_x(\tau_r < \tau_R) = 1 - \left(\frac{x}{R}\right)^{2-\delta}$$

$$\lim_{R\to\infty} \lim_{r\to 0} P_x(\tau_r < \tau_R) = 1.$$

4.9 The process in question is:

$$\begin{cases} dX(t) = \alpha X(t)\,dt + \beta X(t)\,dB(t) \\ X(0) = x \end{cases}$$

i.e. geometric Brownian Motion:

$$X(t) = x \exp\left\{\beta B(t) + \left(\alpha - \frac{\beta^2}{2}\right)t\right\}$$

then apply KBE + distribution of BM.

4.10 Assume first that $\mathcal{G}$ is discrete with partition elements $(G_k)_{k=1,2,\dots}$

Then

$$E_Q(X|G_k) = \frac{\int_{G_k} X(\omega)\, dQ(\omega)}{\int_{G_k} dQ(\omega)} =$$

$$= \frac{\int_{G_k} X(\omega)\, g(\omega)\, dP(\omega)}{\int_{G_k} g(\omega)\, dP(\omega)} \cdot \frac{P(G_k)}{P(G_k)}$$

$$= \frac{E_P(X \cdot g \mid G_k)}{E_P(g \mid G_k)} \quad \dots$$

General case (Conditional expectation a la Kolmogorov)

let $Z \in \mathcal{L}^\infty(\Omega, \mathcal{G}, \mathbb{P})$ (bdd, $\mathcal{G}$-measurable)

$$\mathbb{E}_{\mathbb{Q}}(Z \cdot \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G})) = \mathbb{E}_{\mathbb{Q}}(ZX)$$

by def. of conditional expectation

$$= \mathbb{E}_{\mathbb{P}}(\varrho Z \cdot X)$$

$$\mathbb{E}_{\mathbb{Q}}\left(Z \cdot \frac{\mathbb{E}_{\mathbb{P}}(\varrho X|\mathcal{G})}{\mathbb{E}_{\mathbb{P}}(\varrho|\mathcal{G})}\right) = \mathbb{E}_{\mathbb{P}}\left(\varrho Z \frac{\mathbb{E}_{\mathbb{P}}(\varrho X|\mathcal{G})}{\mathbb{E}_{\mathbb{P}}(\varrho|\mathcal{G})}\right)$$

$$= \mathbb{E}_{\mathbb{P}}\left(\varrho \frac{\mathbb{E}_{\mathbb{P}}(\varrho Z X|\mathcal{G})}{\mathbb{E}_{\mathbb{P}}(\varrho|\mathcal{G})}\right) =$$

$$= \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}(\varrho|\mathcal{G}) \cdot \frac{\mathbb{E}_{\mathbb{P}}(\varrho Z X|\mathcal{G})}{\mathbb{E}_{\mathbb{P}}(\varrho|\mathcal{G})}\right) =$$

$$= \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\varrho Z X|\mathcal{G})) = \mathbb{E}_{\mathbb{P}}(\varrho Z X) \checkmark$$

4.11 $\Omega_n = \{0,1\}^n$ $\quad T=0,\ H=1$

$$\mathbb{P}_n(\omega_1\omega_2\cdots\omega_n) = \left(\frac{2}{3}\right)^{\sum_1^n \omega_k}\left(\frac{1}{3}\right)^{\sum_1^n (1-\omega_k)}$$

$$\mathbb{Q}_n(\omega_1\omega_2\cdots\omega_n) = \left(\frac{1}{2}\right)^n$$

$$\frac{d\mathbb{Q}_n}{d\mathbb{P}_n}(\underline{\omega}) = Z_n(\underline{\omega}) = \left(\frac{3}{2}\right)^n \cdot (2)^{-\sum_1^n \omega_k}$$

$$Z_{m+1} = Z_m \cdot \frac{3}{2} \cdot 2^{-\omega_m}$$

(a) given $(Z_1, Z_2, \ldots Z_m)$, $Z_{m+1} = \begin{cases} Z_m \cdot \frac{3}{2} & \text{with prob. } \frac{1}{3} \\ Z_m \frac{3}{4} & \text{with prob. } \frac{2}{3} \end{cases}$

$\boxed{\text{under } \mathbb{P}}$

(b) $$\mathbb{E}_Q(X|\mathcal{F}_2) = \omega_1 + \omega_2 + \frac{1}{2}$$

$$\cancel{\mathbb{E}_p(X \cdot Z|\mathcal{F}_2) = \mathbb{E}_p\left((\omega_1+\omega_2+\omega_3)\left(\frac{3}{2}\right)^3\right)}$$

$$\mathbb{E}_p\left(X \cdot \frac{Z_3}{Z_2} \middle| \mathcal{F}_2\right) = \mathbb{E}_p\left((\omega_1+\omega_2+\omega_3) \cdot \frac{3}{2} \cdot 2^{-\omega_3} \middle| \mathcal{F}_2\right)$$

$$= \frac{3}{2}(\omega_1+\omega_2)\underbrace{E_P\left(2^{-\omega_3}\middle|\mathcal{F}_2\right)}_{=2/3} +$$

$$\frac{3}{2}\underbrace{E_P\left(\omega_3 2^{-\omega_3}\middle|\mathcal{F}_2\right)}_{=\frac{1}{3}}$$

$$= \omega_1+\omega_2 + \frac{1}{2} \checkmark$$

(c) ... interpretation...

4.12 (a) Follows from Girsanov

$$X(t) := F(t) + B(t)$$

$$E\left(\Phi\left(X(u): 0\leq u\leq T\right)\right) =$$

$$E\left(\Phi\left(B(u): 0\leq u\leq T\right)e^{\int_0^T f(u)dB(u) - \frac{1}{2}\int_0^T (f(u))^2 du}\right)$$

(b) if $\int (f(u))^2 = \infty$ then the R.N derivative doesn't exist

4.14 $Y(t) = B(t) + t$

$M(t) = \exp\left\{-B(t) - \frac{t}{2}\right\}$

$d\mathbb{Q}_t = M(t)\, d\mathbb{P}_t$

$t \mapsto Y(t)$ - is $\boxed{\text{drifted}}$ BM under $\mathbb{P}$

- is standard BM under $\mathbb{Q}$

$\boxed{\mathbb{Q}_\infty \text{ and } \mathbb{P}_\infty \text{ are mutually singular}}$

4.15 $\mathbb{P}(X(t) > M) > 0$

$\Updownarrow$

$\mathbb{Q}(X(t) > M) > 0$

$\|$

$\mathbb{P}(B(t) > M)$