Tools of Modern Probability

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Exercise sheet 1

1. Find all continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ that are rotation invariant and also of product form. That is, there are functions $g : [0, \infty) \to \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$ such that, for every $x, y \in \mathbb{R}$
$$f(x, y) = g(\sqrt{x^2 + y^2}) = u(x)u(y).$$

2. Use the integral substitution $\frac{y^2}{2} := a(x - m)^2$ to show that
$$\int_{-\infty}^{\infty} e^{-a(x-m)^2} \, \mathrm{d}x = \sqrt{\frac{\pi}{a}} \qquad (1)$$
whenever $m \in \mathbb{R}$ and $0 < a \in \mathbb{R}$. We know form class that the value of the integral is $\sqrt{2\pi}$ when $m = 0$ and $a = \frac{1}{2}$.

3. Let $f(x_1, \dots, x_d) = e^{-\frac{x_1^2 + \dots + x_d^2}{2}}$, and let $V = \int_{\mathbb{R}^d} f(\underline{x}) \, \mathrm{d}\underline{x}$.
   - Calculate V using that f is a product:
   $$f(x_1, \dots, x_d) = e^{-\frac{x_1^2}{2}} \cdot e^{-\frac{x_2^2}{2}} \cdot \dots \cdot e^{-\frac{x_d^2}{2}}.$$
   - Write V as a one-dimensional integral using polar coordinate substitution.
   - Compare the two results to get that
   $$c_d = \frac{\sqrt{2\pi}^d}{\int_0^\infty r^{d-1} e^{-\frac{r^2}{2}} \, \mathrm{d}r}.$$

4. Calculate $A_n := \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x$ for every $n = 0, 1, 2, \dots$.

5. Let $B_d \subset \mathbb{R}^d$ be the unit ball is R^d meaning
$$B_d := \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d \middle| x_1^2 + \dots + x_d^2 \le 1 \right\}.$$
(Compare the definition of the sphere – note the inequality here.) Let b_d be the d-dimensional volume of B_d. Calculate b_d.

   (Hint: let $f(r)$ be the surface of the sphere with radius r, and let $g(r)$ be the volume of the ball with radius r.) Convince me (and yourself) that $g'(r) = f(r)$.

6. Try to calculate b_d of the previous exercise the hard way: slice the $d+1$-dimensional sphere into d-dimensional ones to see that
$$b_{d+1} = \int_{-1}^{1} b_d \sqrt{1 - x^2}^d \, \mathrm{d}x.$$

7. For $s > 0$ let
$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, \mathrm{d}x$$
be the Euler gamma function. Check that $\Gamma(s + 1) = s\Gamma(s)$ for all $s > 0$. Check by induction that $\Gamma(n + 1) = n!$ for all $n \in \mathbb{N}$.

8. Calculate $\Gamma\left(\frac{1}{2}\right)$. Express $\Gamma(s)$ for every half-integer $s > 0$ using factorials.

9. Let V be a random vector in $\mathbb{R}^n$ with an n-dimensional standard Gaussian distribution, meaning that it has density

$$f(v_1,\ldots,v_n)=\frac{1}{\sqrt{2\pi}^n}e^{-\frac{v_1^2+\cdots+v_n^2}{2}}.$$

Think of V as the velocity vector of a particle with mass m, so the energy is $E=\frac{m}{2}V^2$. Calculate the distribution of the random variable E. (Meaning: calculate the distribution function and the density.)

10. The free gas is N particles of mass m locked into a box $\Lambda\subset\mathbb{R}^3$ of volume V, with energy depending on the moments only:

$$H(\underline{q},\underline{p})=\sum_{i=1}^{N}\frac{\vec{p}_i^2}{2m},$$

where $\underline{q}\in\Lambda^N$ is the vector of positions and $\underline{p}=(\vec{p}_1,\ldots,\vec{p}_N)\in\mathbb{R}^{3N}$ is the vector of moments.

The microcanonical phase space $\Omega^{micr}_{N,V,E}$ with energy E is the $\{H=E\}$ surface in $\Lambda^N\times\mathbb{R}^{3N}$. The microcanonical reference measure $\mu^{micr}_{N,V,E}$ is a measure on the phase space, that has density $\frac{1}{|\nabla H|}$ w.r.t. (surface) volume.

Calculate the microcanonical partition function $Z_{micr}(N,V,E):=\frac{1}{N!}\mu^{micr}_{N,V,E}(\Omega^{micr}_{N,V,E})$.

(See the explanation in class for the factor $\frac{1}{N!}$.)

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The canonical phase space $\Omega^{can}_{N,V}$ is $\Lambda^N\times\mathbb{R}^{3N}$. The canonical measure $\mu^{micr}_{N,V,\beta}$ with temperature β is the probability measure on the phase space, that has density $\frac{1}{A_{can}(N,V,\beta)}e^{-\beta H}$ (w.r.t. volume), where $A_{can}(N,V,\beta)$ is a normalizing factor.

Calculate the canonical partition function. $Z_{micr}(N,V,E):=\frac{1}{N!}A_{can}(N,V,\beta)$.