Tools of Modern Probability

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Exercise sheet 2

2.1 Describe the asymptotic behaviour of the integral $I_n := \int_{-1}^{1} \sqrt{1-x^2}^n \, \mathrm{d}x$ as $n \to \infty$.

2.2 Let $f_n(x) = \sqrt{1-x^2}^n$ (for $x \in [-1,1]$), and let $g_n(x) = f_n(a_n x)$, where the scaling factor a_n is chosen appropriately, so that $\int_{\mathbb{R}} g_n$ is about 1. Find the limit $g(x) := lim_{n\to\infty} g_n(x)$.

2.3 Let the random vector $V = (V_1, \dots, V_n) \in \mathbb{R}^n$ be uniformly distributed on the (surface of the) $(n-1)$-dimensional sphere of radius $\sqrt{2nE}$ in $\mathbb{R}^n$. Let f_n denote the density of the first marginal V_1 (which is itself a random variable in $\mathbb{R}$, and, of course, its density depends on n). Calculate $f_n(x)$ for every n. Find the limit $f(x) := \lim_{n\to\infty} f_n(x)$.

2.4 *[DeMoivre-Laplace Central Limit Theorem]* We toss a biased coin (where the probability of "heads" is some $p \in (0,1)$) n times independently. Let $q = 1-p$. Let X be the number of heads we see. So X is binomially distributed with parameters n and p, meaning

$$\mathbb{P}(X = k) = Bin(k; n, p) := \binom{n}{k} p^k q^{n-k} \qquad \text{for } k = 0, 1, \dots, n.$$

It is known that X has expectation $\mathbb{E}X = np$ and standard deviation $DX = \sqrt{VarX} = \sqrt{npq}$, so let $Y := \frac{X-np}{\sqrt{npq}}$ be the normalized version of X (which now has expectation 0 and standard deviation 1). Of course, Y is still a discrete random variable, taking only values from a grid of points which are $\frac{1}{\sqrt{npq}}$ apart.

Let us fix $x \in \mathbb{R}$, and choose $k \in \mathbb{Z}$ such that $x \approx \frac{k-np}{\sqrt{npq}}$ as closely as possible, so k is $np + x\sqrt{npq}$ rounded to the nearest integer. Let

$$f_n(x) := \frac{\mathbb{P}(Y = \frac{k-np}{\sqrt{npq}})}{\frac{1}{\sqrt{npq}}} = \sqrt{npq}\mathbb{P}(X = k)$$

be the logical guess for an "approximate density" of Y at x.

Calculate the limit $f(x) := \lim_{n\to\infty} f_n(x)$.

(Hint: it's a good idea to use that $\frac{k}{np} \to 1$ *and* $\frac{n-k}{nq} \to 1$ *where this is enough. However, when you calculate high powers of these, you will need that* $\frac{k}{np} = 1 + x\sqrt{\frac{q}{np}} + O(\frac{1}{n})$ *and* $\frac{n-k}{np} = 1 - x\sqrt{\frac{p}{nq}} + O(\frac{1}{n})$*.)*

2.5 *[Thermodynamic limit of the free gas, microcanonical version]* In the microcanonical description of the free gas, the (microcanonical) *entropy* of the system is

$$S_{micr}(N, V, E) = \log Z_{micr}(N, V, E),$$

where Z_{micr} is the microcanonical partition function (see Exercise 1.10). Let the system size go to infinity in such a way that the density and the energy density remain constant: $N \to \infty$, $V \to \infty$ and $E \to \infty$, but $\frac{N}{V} = \rho$ and $\frac{E}{V} = e$ are constants. Calculate the limiting entropy-density

$$s_{micr}(\rho, e) := \lim_{N\to\infty, V\to\infty, E\to\infty, \frac{N}{V}=\rho, \frac{E}{V}=e} \frac{S_{micr}(N, V, E)}{V}.$$

2.6 *[Thermodynamic limit of the free gas, canonical version]* In the canonical description of the free gas, the (canonical) *entropy* of the system is

$$S_{can}(N, V, \beta) = \frac{3N}{2} + \log Z_{can}(N, V, \beta),$$

where Z_{can} is the canonical partition function (see Exercise 1.11). Let the system size go to infinity in such a way that the density and the temperature remain constant: $N \to \infty$ and $V \to \infty$, but $\frac{N}{V} = \rho$ and β are constants. Calculate the limiting entropy-density

$$s_{can}(\rho, \beta) := \lim_{N \to \infty, V \to \infty, \frac{N}{V} = \rho} \frac{S_{can}(N, V, \beta)}{V}.$$