Tools of Modern Probability
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Exercise sheet 4

4.1 Let $X = [0, 1]$ and let μ be Lebesgue measure on X. Let $f(x) = x^2$. Describe the measure $f_*\mu$.

4.2 Let $X = \{(a_1, a_2, \dots) \,|\, a_k \in \{0, 1\} \text{ for every } k\}$ be the set of $\{0, 1\}$-sequences. Let μ be the measure on X for which

$$\mu(\{(a_1, a_2, \dots) \in X \,|\, a_1 = b_1, \dots, a_N = b_N\}) = \frac{1}{2^N}$$

for every $b_1, \dots, b_N \in \{0, 1\}$. Let $f : X \to \mathbb{R}$ be defined as

$$f(a_1, a_2, \dots) := \sum_{k=1}^{\infty} \frac{a_k}{2^k}.$$

Describe the measure $f_*\mu$.

4.3 Consider the following measure spaces (X, μ):

I. $X = [0, 1]$, μ is Lebesgue measure.
II. $X = [0, \infty)$, μ is Lebesgue measure.
III. $X = \{1, 2 \dots, N\}$, μ is counting measure.
IV. $X = \{1, 2 \dots\}$, μ is counting measure.

Show examples of functions $f_1, f_2, \dots$ and f from X to $\mathbb{R}$ such that f_n converges to f

a.) almost everywhere, but not in L^1,
b.) in L^1, but not almost everywhere,
c.) in L^1, but not in L^2,
d.) in L^2, but not in L^1.

4.4 The characteristic function of a random variable X is the function $\Psi : \mathbb{R} \to \mathbb{C}$ defined as $\Psi(t) = \mathbb{E}(e^{itX})$. Calculate the characteristic function of

(a) The Bernoulli distribution $B(p)$
(b) The "pessimistic geometric distribution with parameter p" – that is, the distribution μ on $\{0, 1, 2 \dots\}$ with weights $\mu(\{k\}) = (1 - p)p^k$ $(k = 0, 1, 2 \dots)$.
(c) The "optimistic geometric distribution with parameter p" – that is, the distribution ν on $\{1, 2, 3, \dots\}$ with weights $\nu(\{k\}) = (1 - p)p^{k-1}$ $(k = 1, 2 \dots)$.
(d) The Poisson distribution with parameter λ – that is, the distribution η on $\{0, 1, 2 \dots\}$ with weights $\eta(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}$ $(k = 0, 1, 2 \dots)$.
(e) The exponential distribution with parameter λ – that is, the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if not} \end{cases}.$$

4.5 Calculate the characteristic function of the normal distribution $\mathcal{N}(m, \sigma^2)$. (Remember the definition from the old times: $\mathcal{N}(m, \sigma^2)$ is the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)

$$f_{m,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

You can save yourself some paperwork if you only do the calculation for $\mathcal{N}(0, 1)$ and reduce the general case to this using the relation between different normal distributions. You can and should use the fact that

$$\int_{-\infty}^{\infty} f_{m,\sigma^2}(x)\,\mathrm{d}x = 1$$

for every m and σ.

4.6 *Dominated convergence and continuous differentiability of the characteristic function.* The Lebesgue dominated convergence theorem is the following

Theorem 1 (dominated convergence) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_1, f_2, \dots$ measurable real valued functions on Ω which converge to the limit function pointwise, μ-almost everywehere. (That is, $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \Omega$, except possibly for a set of x-es with μ-measure zero.) Assume furthermore that the f_n admit a common* integrable dominating function*: there exists a $g : \Omega \to \mathbb{R}$ such that $|f_n(x)| \le g(x)$ for every $x \in \Omega$ and $n \in \mathbb{N}$, and $\int_\Omega g\,\mathrm{d}\mu < \infty$. Then (all the f_n and also f are integrable and)*

$$\lim_{n\to\infty} \int_\Omega f_n\,\mathrm{d}\mu = \int_\Omega f\,\mathrm{d}\mu.$$

Use this theorem to prove the following

Theorem 2 (differentiability of the characteristic function) *Let X be a real valued random variable, $\psi(t) = \mathbb{E}(e^{itX})$ its characteristic function and $n \in \mathbb{N}$. If the n-th moment of X exists and is finite (i.e. $\mathbb{E}(|X|^n) < \infty$), then ψ is n times continuously differentiable and*

$$\psi^{(k)}(0) = i^k \mathbb{E}(X^k), \quad k = 0, 1, 2, \dots, n.$$

4.7 *Exchangeability of integral and limit.* Consider the sequences of functions $f_n : [0, 1] \to \mathbb{R}$ and $g_n : [0, 1] \to \mathbb{R}$ concerning their pointwise limits and the limits of their integrals. Do there exist integrable functions $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$, such that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for Lebesgue almost every $x \in [0, 1]$? What is $\lim_{n\to\infty} \left(\int_0^1 f_n(x)dx \right)$ and $\lim_{n\to\infty} \left(\int_0^1 g_n(x)dx \right)$? Are the conditions of the dominated and monotone convergence theorems and the Fatou lemma satisfied? If yes, what do these theorems ensure about these specific examples?

(a)

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < 1/n, \\ 2n - n^2 x & \text{if } 1/n \le x \le 2/n, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Write n as $n = 2^k + l$, where $k = 0, 1, 2 \dots$ and $l = 0, 1, \dots, 2^k - 1$ (this can be done in a unique way for every n). Now let

$$g_n(x) = \begin{cases} 1 & \text{if } \frac{l}{2^k} \le x < \frac{l+1}{2^k}, \\ 0 & \text{otherwise.} \end{cases}$$

4.8 *Exchangeability of integrals.* Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 \le x - y \le 1, \\ -1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 < y - x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x,y)dx \right) dy$ and $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x,y)dy \right) dx$. What's the situation with the Fubini theorem?

4.9 For real numbers $a_1, a_2, a_3, \ldots$ define the infinite product $\prod\limits_{k=1}^{\infty} a_k$ as

$$\prod_{k=1}^{\infty} a_k := \lim_{n\to\infty} \prod_{k=1}^{n} a_k,$$

whenever this limit exists.

Let $p_1, p_2, p_3, \ldots$ satisfy $0 \le p_k < 1$ for all k. Show that $\prod\limits_{k=1}^{\infty} (1 - p_k) > 0$ if and only if $\sum\limits_{k=1}^{\infty} p_k < \infty$.

(Hint: estimate the logarithm of $(1 - p)$ *with* p*.)*

4.10 Let $X_1, X_2, \ldots$ be independent random variables such that

$$\mathbb{P}(X_n = n^2 - 1) = \frac{1}{n^2}, \quad \mathbb{P}(X_n = -1) = 1 - \frac{1}{n^2}.$$

Show that $\mathbb{E}X_n = 0$ for every n, but

$$\lim_{n\to\infty} \frac{X_1 + \ldots X_n}{n} = -1$$

almost surely.

4.11 Prove that for *any* sequence $X_1, X_2, \ldots$ of random variables (real valued, defined on the same probability space) there exists a sequence $c_1, c_2, \ldots$ of numbers such that

$$\frac{X_n}{c_n} \to 0 \text{ almost surely.}$$

4.12 Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution *Bernoulli*(p) for some $p \in (0; 1)$ but $p \neq \frac{1}{2}$. Let $Y := \sum_{n=1}^{\infty} 2^{-n} X_n$. (The sum is absolutely convergent.) Show that the distribution of Y is continuous, but singular w.r.t. Lebesgue measure.