

Tools of Modern Probability

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Exercise sheet 5

- 5.1 Let (X, \mathcal{F}) be a measurable space and let μ, ν be σ -finite measures on it. Show that there is a countable partition $X = \dot{\bigcup}_i A_i$ such that $\mu(A_i) < \infty$ and $\nu(A_i) < \infty$ for every i . Use this to show that the special case of the Radon-Nikodym theorem for finite measures implies the general theorem (for σ -finite measures).
- 5.2 Let λ be Lebesgue measure and χ be counting measure on \mathbb{R} (with the Borel σ -algebra). Show that λ does not have a density with respect to χ . (Hint: consider 1-element sets.)
- 5.3 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A \in \mathcal{F}$. Define $X : \Omega \rightarrow \mathbb{R}$ as $X(\omega) = \mathbf{1}_A(\omega)$ and let $\mu = X_*\mathbb{P}$ be the distribution of X . Show that μ is absolutely continuous w.r.t counting measure, show that it also has a density. What is the density?
- 5.4 Let X be a discrete random variable and let μ be its distribution. Give the density of μ w.r.t. counting measure.
- 5.5 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X : \Omega \rightarrow \mathbb{R}^+$ be integrable and let $\mathcal{G} \subset \mathcal{F}$ be a sub- σ -algebra. Define $\nu : \mathcal{G} \rightarrow \mathbb{R}^+$ by $\nu(A) := \int_A X d\mathbb{P}$ (whenever $A \in \mathcal{G}$). Check that ν is a measure on (Ω, \mathcal{G}) .
- 5.6 Let X be a nonempty set and let $\mathcal{F}_i \subset 2^X$ be a σ -algebra for every $i \in I$, where I is some index set. I may be arbitrary (possibly much bigger than countable), but we assume $I \neq \emptyset$. Show that $\mathcal{F} := \bigcap_{i \in I} \mathcal{F}_i$ is also a σ -algebra. (Note that the assumption $I \neq \emptyset$ is important.)
- 5.7 Let (Ω, \mathcal{F}) be a measurable space and let $X : \Omega \rightarrow \mathbb{R}$ be (Borel-)measurable. Let $(\mathcal{G}_i)_{i \in I}$ be the family of all σ -algebras over Ω such that X is \mathcal{G}_i -measurable, and let $\mathcal{G} := \bigcap_{i \in I} \mathcal{G}_i$. Show that \mathcal{G} is the *smallest* σ -algebra for which X is measurable. (In what sense exactly is it the smallest?)
- 5.8 Let (Ω, \mathcal{F}) be a measurable space, let $X : \Omega \rightarrow \mathbb{R}$ be $(\mathcal{F}, \mathcal{B})$ -measurable, where \mathcal{B} is the Borel σ -algebra on \mathbb{R} . Let $\sigma(X)$ be the smallest σ -algebra on Ω for which X is measurable. (This exists by the previous exercise.) This is called the σ -algebra generated by X . Show that
- $$\sigma(X) = \{X^{-1}(B) \mid B \in \mathcal{B}\}.$$
- 5.9 Let (Ω, \mathcal{F}) be a measurable space, and let $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{F}$ be sub- σ -algebras. We say that \mathcal{G}_1 and \mathcal{G}_2 are independent if any $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$ are independent. Show that if the random variables X and Y are independent, then $\sigma(X)$ and $\sigma(Y)$ are independent.
- 5.10 Let (Ω, \mathcal{F}) be a measurable space, and let $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{F}$ be sub- σ -algebras. Let X and Y be random variables, $X \in \mathcal{G}_1, Y \in \mathcal{G}_2$. Show that if \mathcal{G}_1 and \mathcal{G}_2 are independent, then X and Y are independent.
- 5.11 Show that if X is a random variable, $f : \mathbb{R} \rightarrow \mathbb{R}$ measurable and $Y = f(X)$, then $\sigma(Y) \subset \sigma(X)$. Show an example when equality holds, and an example when not.
- 5.12 Show that if X, Y are independent random variables and $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are measurable, then $f(X)$ and $g(Y)$ are also independent.
- 5.13 Show that the random variables $X, Y : \Omega \rightarrow \mathbb{R}$ are independent if and only if the (joint) distribution of the pair (X, Y) (which is a probability measure on \mathbb{R}^2) is the product of the distributions of X and Y .

- 5.14 Show that if X and Y are independent and integrable, then $\mathbb{E}(XY) = \mathbb{E}X\mathbb{E}Y$.
- 5.15 Show that if the random variable X is independent of the σ -algebra \mathcal{G} , then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}X$.
- 5.16 Let $\Omega = \{a, b, c\}$ and \mathbb{P} the uniform measure on it. Let $X = \mathbf{1}_{\{c\}}$ and let $\mathcal{G} = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$. Calculate $\mathbb{E}(X|\mathcal{G})$.
- 5.17 We roll two fair dice and let X, Y be the numbers rolled. Calculate $\mathbb{E}(X|X + Y)$.
- 5.18 Let $\Omega = [0, 1]^2$ and let \mathbb{P} be Lebesgue measure on Ω . Let $X, Y : \Omega \rightarrow \mathbb{R}$ be defined as $X(u, v) = u$ and $Y(u, v) = \sqrt{u + v}$. Calculate $\mathbb{E}(Y|X)$.
- 5.19 Let U and V be independent random variables, uniformly distributed on $[0, 1]$. Calculate $\mathbb{E}(\sqrt{U + V}|U)$.
- 5.20 Let U and V be independent random variables, uniformly distributed on $[0, 1]$. Calculate $\mathbb{E}(U + V|U - V)$.
- 5.21 Let U and V be independent random variables, uniformly distributed on $[0, 1]$. Calculate $\mathbb{E}(\sqrt{U + V}|U - V)$.
- 5.22 Let X and Y be independent standard Gaussian random variables. Let $U = X + Y$ and $V = 2X - Y$. Calculate $\mathbb{E}(V|U)$. (*Hint: if W is independent of U , then $\mathbb{E}(W|U) = \mathbb{E}W$. If you choose $\lambda \in \mathbb{R}$ cleverly, then $W := V - \lambda U$ will be independent of U . (Since U and W are jointly Gaussian, to show independence it's enough to check that $\text{Cov}(U, W) = 0$.) Then write $V = \lambda U + W$.)*