Tools of Modern Probability
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Exercise sheet 1, fall 2020

1.1 Find all continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ that are rotation invariant and also of product form. That is, there are functions $g : [0, \infty) \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that, for every $x, y \in \mathbb{R}$

$$f(x, y) = g(\sqrt{x^2 + y^2}) = h(x)h(y).$$

(Hint: write everything as the function of the ***square*** *of the radius, e.g. by defining* $u := x^2$, $v := y^2$ *and* $G(z) := g(\sqrt{z})$. *Then you should get* $G(u + v) = const G(u)G(v)$. *Now study the logarithm of* G.*)*

1.2 Use the integral substitution $\frac{y^2}{2} := a(x - m)^2$ to show that

$$\int_{-\infty}^{\infty} e^{-a(x-m)^2} \, \mathrm{d}x = \sqrt{\frac{\pi}{a}} \tag{1}$$

whenever $m \in \mathbb{R}$ and $0 < a \in \mathbb{R}$. We know form class that the value of the integral is $\sqrt{2\pi}$ when $m = 0$ and $a = \frac{1}{2}$.

1.3 Let $f(x_1, \dots, x_d) = e^{-\frac{x_1^2 + \dots + x_d^2}{2}}$, and let $V = \int_{\mathbb{R}^d} f(\underline{x}) \, \mathrm{d}\underline{x}$.

- Calculate V using that f is a product:

$$f(x_1, \dots, x_d) = e^{-\frac{x_1^2}{2}} \cdot e^{-\frac{x_2^2}{2}} \cdot \dots \cdot e^{-\frac{x_d^2}{2}}.$$

- Write V as a one-dimensional integral using polar coordinate substitution.
- Compare the two results to get that

$$c_d = \frac{\sqrt{2\pi}^d}{\int_0^\infty r^{d-1} e^{-\frac{r^2}{2}} \, \mathrm{d}r}.$$

1.4 Calculate $A_n := \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x$ for every $n = 0, 1, 2, \dots$ the hard way: if $n \geq 2$, then

$$A_n = \int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \cos^{n-2} x \, \mathrm{d}x = A_{n-2} - \int_0^{\frac{\pi}{2}} [\sin x] \left[\sin x \cos^{n-2} x\right] \, \mathrm{d}x,$$

and you can use integration by parts in the second term.

1.5 Let $B_d \subset \mathbb{R}^d$ be the unit ball in R^d meaning

$$B_d := \left\{(x_1, \dots, x_d) \in \mathbb{R}^d \middle| x_1^2 + \dots + x_d^2 \leq 1\right\}.$$

(Compare the definition of the sphere – note the inequality here.) Let b_d be the d-dimensional volume of B_d. Calculate b_d.

(Hint: the volume is the integral of the indicator function. Use the theorem about polar coordinate substitution in d dimensions.)

1.6 Try to calculate b_d of the previous exercise the hard way: slice the $d+1$-dimensional sphere into d-dimensional ones to see that

$$b_{d+1} = \int_{-1}^{1} b_d \sqrt{1 - x^2}^d \, \mathrm{d}x.$$

1.7 For $s > 0$ let

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, \mathrm{d}x$$

be the Euler gamma function. Check that $\Gamma(s+1) = s\Gamma(s)$ for all $s > 0$. Check by induction that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

1.8 Calculate $\Gamma\left(\frac{1}{2}\right)$. Express $\Gamma(s)$ for every half-integer $s > 0$ using factorials.

1.9 Fix some $s, t > 0$. Consider $f : (0,\infty) \times (0,\infty) \to \mathbb{R}$ defined by $f(x,y) := x^{s-1}e^{-x}y^{t-1}e^{-y}$ (for all $x, y > 0$). Calculate $\int_{(0,\infty)^2} f(x,y) \, \mathrm{d}x \, \mathrm{d}y$ in two different ways:

a.) By using that f has product form,

b.) using the substitution $u := x + y$, $\xi := \frac{y}{x+y}$. *(If it's easier, you can do this in two steps: first $u := x + y$, $v := y$; second $\xi := v/u$.)*

Comparing the two results, express the Beta function $B(s,t) := \int_0^1 (1-\xi)^{s-1}\xi^{t-1} \, \mathrm{d}\xi$ using the Euler gamma function.

1.10 Calculate $A_n := \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x$ for every $n = 0, 1, 2, \ldots$ using the substitution $\xi := \cos x$ and the result of the previous exercise.

1.11 Describe the asymptotic behaviour of the integral $I_n := \int_{-1}^1 \sqrt{1-x^2}^n \, \mathrm{d}x$ as $n \to \infty$.

1.12 Describe the asymptotic behaviour of the integral $I_n := \int_{-2}^2 \sqrt{4-x^2}^n \, \mathrm{d}x$ as $n \to \infty$.

1.13 Let

$$f_n(x) = \begin{cases} \cos^n x & \text{if } x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 0 & \text{if not} \end{cases}.$$

Let $g_n(x) = f_n(v_n x)$, where the scaling factors v_n are chosen appropriately, so that $\int_{\mathbb{R}} g_n \to 1$ *(More precisely: g_n should be integrated on all of its domain.)* Find the limit $g(x) := lim_{n\to\infty} g_n(x)$.

1.14 Let $f_n(x) = \sqrt{4-x^2}^n$ (for $x \in [-2,2]$), and let $g_n(x) = u_n f_n(v_n x)$, where the scaling factors u_n and v_n are chosen appropriately, so that $g_n(0) \to 1$ and $\int_{\mathbb{R}} g_n \to 1$ *(More precisely: g_n should be integrated on all of its domain.)* Find the limit $g(x) := lim_{n\to\infty} g_n(x)$.

1.15 Let $a < 0 < b$ and let $h : [a,b] \to \mathbb{R}$ be twice differentiable with a unique non-degenerate local maximum at 0. Denote $A := h(0)$ and $B := -h''(0)$. Let $f_n : [a,b] \to \mathbb{R}$ with $f_n(x) = e^{nh(x)}$. Now let $u_n > 0$ and $v_n > 0$ be two sequences of scaling factors, and define g_n as

$$g_n(x) := u_n f_n(v_n x),$$

for the $x \in \mathbb{R}$ where this makes sense. (This means stretching the graph of f_n vertically with a factor u_n and shrinking it horizontally with a factor v_n.)

a.) How should we choose u_n to make sure that $g_n(0) \to 0$ as $n \to \infty$? *(Of course, there are many such sequences: if u_n works and $\bar{u}_n \sim u_n$, then $\bar{u}_n$ works as well. So give a simple example.)*

b.) Fix u_n as in the previous part. Now how should be choose v_n to make sure that

$$\int_{D_n} g_n(x) \, \mathrm{d}x \to 1$$

as $n \to \infty$? (Here let D_n denote the domain of g_n.)

c.) With u_n and v_n chosen as above, calculate $g(x) := \lim_{n\to\infty} g_n(x)$ for all $x \in \mathbb{R}$.

1.16 Let the random vector $V = (V_1, \dots, V_n) \in \mathbb{R}^n$ be uniformly distributed on the (surface of the) $(n-1)$-dimensional sphere of radius $\sqrt{2nE}$ in $\mathbb{R}^n$. Let f_n denote the density of the first marginal V_1 (which is itself a random variable in $\mathbb{R}$, and, of course, its density depends on n). Calculate $f_n(x)$ for every n. Find the limit $f(x) := \lim_{n\to\infty} f_n(x)$.

1.17 *[DeMoivre-Laplace Central Limit Theorem]* We toss a biased coin (where the probability of "heads" is some $p \in (0,1)$) n times independently. Let $q = 1 - p$. Let X be the number of heads we see. So X is binomially distributed with parameters n and p, meaning

$$\mathbb{P}(X = k) = Bin(k; n, p) := \binom{n}{k} p^k q^{n-k} \quad \text{for } k = 0, 1, \dots, n.$$

It is known that X has expectation $\mathbb{E}X = np$ and standard deviation $DX = \sqrt{VarX} = \sqrt{npq}$, so let $Y := \frac{X-np}{\sqrt{npq}}$ be the normalized version of X (which now has expectation 0 and standard deviation 1). Of course, Y is still a discrete random variable, taking only values from a grid of points which are $\frac{1}{\sqrt{npq}}$ apart.

Let us fix $x \in \mathbb{R}$, and choose $k \in \mathbb{Z}$ such that $x \approx \frac{k-np}{\sqrt{npq}}$ as closely as possible, so k is $np + x\sqrt{npq}$ rounded to the nearest integer. Let

$$f_n(x) := \frac{\mathbb{P}(Y = \frac{k-np}{\sqrt{npq}})}{\frac{1}{\sqrt{npq}}} = \sqrt{npq}\,\mathbb{P}(X = k)$$

be the logical guess for an "approximate density" of Y at x.

Calculate the limit $f(x) := \lim_{n\to\infty} f_n(x)$.

Hint:

Use Stirling's approximation $n! \sim \frac{n^n\sqrt{2\pi n}}{e^n}$, and the fact that $k = np + x\sqrt{npq} + \Delta$, where $\Delta = \Delta(n,x) \in [-\frac{1}{2}, \frac{1}{2}]$, so $\Delta = O(1)$. Use this in the following forms:

$$k = np + x\sqrt{npq} + \Delta \quad , \quad n - k = nq - x\sqrt{npq} - \Delta \tag{2}$$

$$\frac{k}{np} = 1 + x\sqrt{\frac{q}{np}} + \frac{\Delta}{np} \quad , \quad \frac{n-k}{nq} = 1 - x\sqrt{\frac{p}{nq}} - \frac{\Delta}{nq} \tag{3}$$

$$\frac{k}{np} = 1 + o(1) \quad , \quad \frac{n-k}{nq} = 1 + o(1) \tag{4}$$

Notice that (2) is a bit stronger than if we only wrote $k = np + x\sqrt{npq} + O(1)$ and $n - k = nq - x\sqrt{npq} + O(1)$. This will be important, since Δ will cancel out at some point.

At some point the calculation may become more transparent if you calculate the logarithm of $f_n(x)$.