Tools of Modern Probability

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Exercise sheet 3, fall 2019

3.1 *Weak convergence and densities.* Prove the following

Theorem 1 *Let $\mu_1, \mu_2, \dots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continouos w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \dots$ and f, respectively. Denote their distribution functions by $F_1, F_2, \dots$ and F, respectively. Suppose that $f_n(x) \overset{n\to\infty}{\longrightarrow} f(x)$ for every $x \in \mathbb{R}$. Then $F_n(x) \overset{n\to\infty}{\longrightarrow} F(x)$ for every $x \in \mathbb{R}$.*

(Hint: Use the Fatou lemma to show that $F(x) \le \liminf_{n\to\infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.)

3.2 Which of the spaces V below are linear spaces and why?

a.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 + 2x_2 = 0\}$, with the usual addition and the usual multiplication by a scalar.

b.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 + 2x_2 = 3\}$, with the usual addition and the usual multiplication by a scalar.

c.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 \ge 0\}$, with the usual addition and the usual multiplication by a scalar.

d.) $V := \{f : (0,1) \to \mathbb{R} \,|\, f$ is continuous and $|f| \le 100\}$, with the usual addition and the usual multiplication by a scalar.

e.) $V := \{f : (0,1) \to \mathbb{R} \,|\, f$ is continuous and bounded$\}$, with the usual addition and the usual multiplication by a scalar.

3.3 On the linear spaces V and W below, which of the given transformations $T : V \to W$ are linear and why?

a.) $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $T((x_1, x_2, x_3)) := (x_1, x_2 + x_3)$.

b.) $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $T((x_1, x_2, x_3)) := (x_1, 1 + x_3)$.

c.) $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $T((x_1, x_2, x_3)) := (x_1, x_2 x_3)$.

d.) $V := \{f : (-1,1) \to \mathbb{R} \,|\, f$ differentiable$\}$, with the usual addition and the usual multiplication by a scalar; $W := \mathbb{R}$; $T(f) := f'(0)$.

3.4 On the linear spaces V below, which of the given two-variable functions $B : V \to \mathbb{R}$ are bilinear forms? Which ones are symmetric and positive definite? Why?

a.) $V = \mathbb{R}^3$, $B((x_1, x_2, x_3), (y_1, y_2, y_3)) := x_1 y_2 + x_2 y_3 + x_3 y_1$

b.) $V = \mathbb{R}^2$, $B((x_1, x_2), (y_1, y_2)) := x_1 x_2 + y_1 y_2$

c.) $V = \mathbb{R}^2$, $B((x_1, x_2), (y_1, y_2)) := x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2$

d.) $V := \{f : [-1,1] \to \mathbb{R} \,|\, f$ is differentiable$\}$, with the usual addition and the usual multiplication by a scalar; $B(f,g) := \int_{-1}^{1} x^2 f(x) g(x) \,\mathrm{d}x$

e.) $V := \{f : [-1,1] \to \mathbb{R} \,|\, f$ is differentiable$\}$, with the usual addition and the usual multiplication by a scalar; $B(f,g) := \int_{-1}^{1} x f(x) g(x) \,\mathrm{d}x$

f.) $V := \{f : [-1,1] \to \mathbb{R} \,|\, f$ is differentiable$\}$, with the usual addition and the usual multiplication by a scalar; $B(f,g) := \int_{-1}^{1} f'(x) g(x) \,\mathrm{d}x$

3.5 Let V be an inner product space. Show that the function $N : V \to \mathbb{R}$ defined as $N(x) := \sqrt{\langle x, x\rangle}$ is indeed a norm (usually denoted as $||x|| = N(x)$).

3.6 Let V be an inner product space, and let d denote the natural metric on it (defined as $d(x, y) := ||x - y||$). Let $x \in V$, let $D \subset V$ be convex, and assume that $d(x, D) = R > 0$ (where $d(x, D) := \inf\{d(x, y) \,|\, y \in D\}$ is the distance of x and D). Find a number $C \in \mathbb{R}$ (possibly depending on R) such that if $u, v \in D$, $d(x, u) \le R + \varepsilon$ and $d(x, v) \le R + \varepsilon$ with some $\varepsilon < R$, then $d(u, v) \le C\sqrt{\varepsilon}$. *(Hint: estimate the length of the longest line segment that fits in the shell* $\{y \in V \mid R \le d(x, y) \le R + \varepsilon\}$. *A two-dimensional drawing will help.)*

3.7 Let V be an inner product space, and let d denote the natural metric (defined as $d(x, y) := ||x - y||$).

a.) Let $a, c, x \in V$ with $x \neq c$. Calculate the distance of a from the line $\{c + t(x - c) \,|\, t \in \mathbb{R}\}$ using $||a - c||$, $||x - c||$ and $\langle a - c, x - c\rangle$.

b.) Let $E \subset V$ be a linear subspace and let $a \in V$. Suppose that $c \in E$ is such that $d(a, x) \ge d(a, c)$ for every $x \in E$ – which means that c is the point in E which is closest to a. Prove that E is orthogonal to $a - c$, meaning that $\langle x, a - c\rangle = 0$ for every $x \in E$.

3.8 Let V be an inner product space over $\mathbb{R}$ and let $f : V \to \mathbb{R}$ be a linear form. Let $E := \{y \in V \,|\, f(y) = 0\}$ be the null-space of f. Suppose that $f(a) = 1$, $c \in E$ and $a - c$ is orthogonal to E, meaning $(a - c)y = 0$ for every $y \in E$. Now, for any $x \in V$, find the $\lambda \in \mathbb{R}$ for which $x_1 := x - \lambda(a - c) \in E$. Use this to get the relation between $f(x)$ and $(a - c)x$.

3.9 Represent the following functions $f : V \to \mathbb{R}$ as multiplication by a fixed vector, whenever this is possible due to the Riesz representation theorem.

a.) $V = \mathbb{R}^{10}$ with the usual inner product, $f((x_1, \dots, x_{10})) := x_5$ (evaluation at 5)

b.) $V = \mathbb{R}^{10}$ with the usual inner product, $f((x_1, \dots, x_{10})) := x_6 - x_5$ (discrete derivative at 5).

c.) $V = \mathbb{R}^{10}$ with the usual inner product, $f((x_1, \dots, x_{10})) := x_6 - 2x_5 + x_4$ (discrete second derivative at 5).

d.) $V = l^2 := \{x : \mathbb{N} \to \mathbb{R} \,|\, \sum_{i=1}^{\infty} x^2(i) < \infty\}$, with the inner product $x \cdot y := \sum_{i=1}^{\infty} x(i)y(i)$; $f(x) := \sum_{i=1}^{100} x(i)$.

e.) $V = l^2 := \{x : \mathbb{N} \to \mathbb{R} \,|\, \sum_{i=1}^{\infty} x^2(i) < \infty\}$, with the inner product $x \cdot y := \sum_{i=1}^{\infty} x(i)y(i)$; $f(x) := \sum_{i=1}^{\infty} x(i)$.

f.) $V = l^2 := \{x : \mathbb{N} \to \mathbb{R} \,|\, \sum_{i=1}^{\infty} x^2(i) < \infty\}$, with the inner product $x \cdot y := \sum_{i=1}^{\infty} x(i)y(i)$; $f(x) := \sum_{i=1}^{\infty} x^2(i)$.

g.) $V = L^2([0, 1]) := \{x : [0, 1] \to \mathbb{R} \,|\, \int_0^1 x^2(t)\, \mathrm{d}t < \infty\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\, \mathrm{d}t$; $f(x) := x(\frac{1}{2})$ (evaluation at $\frac{1}{2}$).

h.) $V = L^2([0, 1]) := \{x : [0, 1] \to \mathbb{R} \,|\, \int_0^1 x^2(t)\, \mathrm{d}t < \infty\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\, \mathrm{d}t$; $f(x) := x'(\frac{1}{2})$ (derivative at $\frac{1}{2}$).

i.) $V = L^2([0, 1]) := \{x : [0, 1] \to \mathbb{R} \,|\, \int_0^1 x^2(t)\, \mathrm{d}t < \infty\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\, \mathrm{d}t$; $f(x) := \int_{0.2}^{0.7} x(t)\, \mathrm{d}t$.

j.) $V = \{x : [0, 1] \to \mathbb{R} \,|\, \int_0^1 x^2(t)\, \mathrm{d}t < \infty,\ f \text{ is differentiable}\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\, \mathrm{d}t$; $f(x) := x'(\frac{1}{2})$.

k.) $V = \{x : [0, 1] \to \mathbb{R} \,|\, \int_0^1 x^2(t)\, \mathrm{d}t < \infty,\ f \text{ is continuous}\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\, \mathrm{d}t$; $f(x) := x(\frac{1}{2})$.

l.) $V = \{x : [0, 1] \to \mathbb{R} \,|\, \int_0^1 x^2(t)\, \mathrm{d}t < \infty,\ f \text{ is continuous}\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\, \mathrm{d}t$; $f(x) := \int_{0.2}^{0.7} x(t)\, \mathrm{d}t$.

3.10 Let $(X, \mathcal{F})$ be a measurable space and let μ, ν be σ-finite measures on it. Show that there is a countable partition $X = \dot{\bigcup}_i A_i$ such that $\mu(A_i) < \infty$ and $\nu(A_i) < \infty$ for every i. Use this to show that the special case of the Radon-Nikodym theorem for finite measures implies the general theorem (for σ-finite measures).

3.11 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X : \Omega \to \mathbb{R}^+$ be integrable and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Define $\nu : \mathcal{G} \to \mathbb{R}^+$ by $\nu(A) := \int_A X \,\mathrm{d}\mathbb{P}$ (whenever $A \in \mathcal{G}$). Check that ν is a measure on $(\Omega, \mathcal{G})$.

3.12 Let X be a nonempty set and let $\mathcal{F}_i \subset 2^X$ be a σ-algebra for every $i \in I$, where I is some index set. I may be arbitrary (possibly much bigger that countable), but we assume $I \neq \emptyset$. Show that $\mathcal{F} := \bigcap_{i \in I} \mathcal{F}_i$ is also a σ-algebra. (Note that the assumption $I \neq \emptyset$ is important.)

3.13 Let $(\Omega, \mathcal{F})$ be a probability space and let $X : \Omega \to \mathbb{R}$ be (Borel-)measurable. Let $(\mathcal{G}_i)_{i \in I}$ be the family of all σ-algebras over Ω such that X is $\mathcal{G}_i$-measurable, and let $\mathcal{G} := \bigcap_{i \in I} \mathcal{G}_i$. Show that $\mathcal{G}$ is the *smallest* σ-algebra for which X is measurable. (In what sense exactly is it the smallest?)

3.14 Let $(\Omega, \mathcal{F})$ be a probability space, let $X : \Omega \to \mathbb{R}$ be $(\mathcal{F}, \mathcal{B})$-measurable, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. Let $\sigma(X)$ be the smallest σ-algebra on Ω for which X is measurable. (This exists by the previous exercise.) This is called the *σ-algebra generated by X*. Show that

$$\sigma(X) = \{X^{-1}(B) \mid B \in \mathcal{B}\}.$$

3.15 Let $\Omega = \{a, b, c\}$ and $\mathbb{P}$ the uniform measure on it. Let $X = \mathbf{1}_{\{c\}}$ and let $\mathcal{G} = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$. Calculate $\mathbb{E}(X|\mathcal{G})$.

3.16 We roll two fair dice and let X, Y be the numbers rolled. Calculate $\mathbb{E}(X|\sigma(X + Y))$. For this you may want to introduce a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

3.17 Let $\Omega = [0, 1]^2$ and let $\mathbb{P}$ be Lebesgue measure on Ω. Let $X, Y : \Omega \to \mathbb{R}$ be defined as $X(u, v) = u$ and $Y(u, v) = \sqrt{u + v}$. Calculate $\mathbb{E}(Y|\sigma(X))$.

3.18 Let U and V be independent random variables, uniformly distributed on $[0, 1]$. Find a function $h : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}(\sqrt{U + V}|\sigma(U)) = h(U)$. (Note that both sides of the equation are random variables.)