Riesz representation theorem

Could be any normed space here

Consider a Hilbert space V and a linear form $f: V \to \mathbb{R}$.

<u>Def</u>: f is <u>continuous</u> if for any convergent sequence
$v_1, v_2, \ldots \in V$, $f(\lim_{n\to\infty} v_n) = \lim_{n\to\infty} f(v_n)$.

[There is nothing new in this: this is ~~the~~ (1 possible) definition of continuity in any metric (or even topological) space.]

<u>Def</u>: f is <u>bounded</u> if $\exists K < \infty$ s.t.

$$|f(v)| \leq K \|v\| \quad \text{for every } v \in V.$$

[Here, of course $\|v\| = \sqrt{\langle v, v\rangle}$, the induced norm.]

<u>Important</u>: This is NOT the same as boundedness of f as a function:

we do <u>not</u> require $|f(v)| \leq K$ for every $v \in V$,

only for $v \in V$ with $\|v\| \leq 1$.

That is: boundedness as a linear form

$$\Updownarrow \text{def}$$

boundedness as a function <u>on the unit ball</u>.

[Easy HW: If a linear form f is bounded <u>as a function</u>, then $f \equiv 0$ — pretty boring.]

<u>Thm:</u> A linear form is bounded if and only if it is continuous.

<u>Proof:</u>

1.) Bounded $\Rightarrow$ Continuous: $v_n \to v$ means $\|v_n - v\| \to 0$.

~~Then by boundedness~~

Then

$$|f(v_n) - f(v)| \overset{\text{linearity}}{=} |f(v_n - v)| \overset{\text{boundedness}}{\leq} K\|v_n - v\| \to 0.$$

2.) Continuity $\Rightarrow$ boundedness:

Assume that f is <u>not</u> bounded, so $\forall n \in \mathbb{N}\ \exists v_n \in V$ s.t. $|f(v_n)| > n\|v_n\|$. Then <u>~~of course~~ by linearity</u> any $\lambda \cdot v_n$ will also do, so pick ~~$\tilde{v}_n := \frac{v_n}{|f(v_n)|}$ to make sure~~ that

~~$\|f(\tilde{v}_n)\|$ $f(\tilde{v}_n$~~ $\tilde{v}_n := \frac{v_n}{f(v_n)}$ to make sure

that $f(\tilde{v}_n) = \frac{1}{f(v_n)} f(v_n) = 1 \not\to 0$,

while $\|\tilde{v}_n\| = \frac{1}{\|f(v_n)\|}\|v_n\| < \frac{\|v_n\|}{n\|v_n\|} < \frac{1}{n} \to 0$

so $\tilde{v}_n \to \underline{0}$, but $f(\tilde{v}_n) \not\to 0 = f(\underline{0})$, so f is not continuous. $\square$

[<u>Essence:</u> there are <u>many</u> directions, and if expansion is huge in some directions, this is what spoils continuity]

Example:

1.) Let $a \in V$ be fixed and define $f: V \to V$ as

$$f(v) := \langle a, v \rangle \text{ for all } v \in V,$$

So f = scalar multiplication by a.

This is a linear form — of course, $f = \langle a, \cdot \rangle$ so it's a linear form by the def of the bilinear form $\langle \cdot, \cdot \rangle$.

This f is bounded:

$$|f(v)| = |\langle a, v \rangle| \underset{\substack{\text{Schwartz} \\ \text{inequality}}}{\overset{\text{Cauchy-}}{\leq}} \|a\| \|v\|, \text{ So } K := \|a\| \text{ will do.}$$

2.) In $V = \mathbb{R}^n$ every linear form is like that:

if $f: \mathbb{R}^n \to \mathbb{R}$ is linear, then let

$$\cancel{e_1 := (1, 0, \dots, 0)^T} \; e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

be the standard basis, and choose

$$\mathbb{R}^n \ni a := \begin{pmatrix} f(e_1) \\ f(e_2) \\ \vdots \\ f(e_n) \end{pmatrix}. \text{ Then } v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

(coordinate representation),

clever def. of "a"

$$\text{so } f(v) \overset{\text{linearity}}{=\!=} x_1 f(e_1) + x_2 f(e_2) + \dots + x_n f(e_n) = a_1 x_1 + \dots + a_n x_n = \langle a, v \rangle.$$

So in $\mathbb{R}^n$, <u>every</u> linear form is bounded.

<u>Question 1</u>: Can anyone find <u>another</u> example of a bounded linear form (on a Hilbert space)?

[By the way: Can anyone find another example of ~~a bounded~~ <u>an arbitrary</u> linear form on a Hilbert space? (Question 2)]

<u>Answer 1</u>: No. [Answer 2: Yes, although---]

<u>Theorem</u> For any bounded linear form $f: V \to \mathbb{R}$ on a Hilbert space V, $\exists!\ a \in V$ s.t. $f(v) = \langle a, v \rangle$ for all $v \in V$.

RIESZ REPRESENTATION THEOREM

This theorem is <u>very useful</u> - we will discuss 1 application.

[<u>Remark</u>: FACT (highly non-trivial):
On any infinite dimensional Hilbert space there exists a non-bounded linear form.
<u>Proof</u>: easy HW, once you accept the following (highly non-trivial) theorem:
<u>Thm</u>: Every linear space has a (Hamel) basis.
<u>Def</u>: A family of vectors $U \subset V$ is the <u>basis</u> (or: Hamel basis) of the linear space V, if every $v \in V$ can be

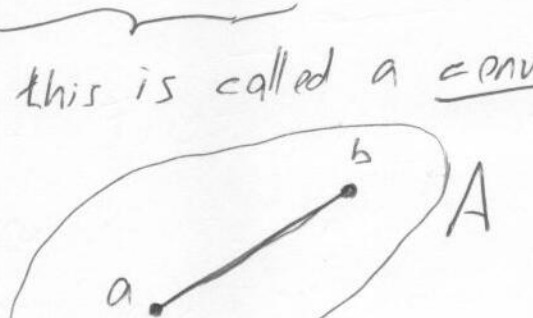
[Uniquely written as a *finite* linear combination of vectors in $\mathcal{U}$.]

[*Proof* of existence of a Hamel basis:
abstract set theory, using the axiom of choice.]

To prove the Riesz representation thm, we start with a seemingly unrelated / uninteresting geometrical fact.

Def: Let V be a linear space and $A \subset V$, a subset. The subset $A \subset V$ is called *convex*, if for any $a, b \in A$ and $t \in [0,1]$,

$$\underbrace{(1-t)a + tb}_{\text{this is called a convex (linear) combination of } a \text{ and } b} \in A \text{ as well.}$$



[$t \mapsto (1-t)a + tb$ is just a parametrization of the line segment between a and b: for $t=0$ it gives a, $t=1$ ~ b, and it's "straight".]

In words: convex means "no hide-and-seek possible": from any point $a \in A$, any other point $b \in A$ is visible (the line segment is in A).

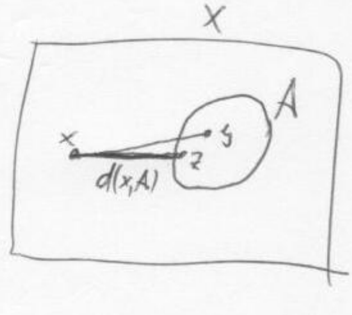
~~Def~~: Let (X,d) be a metric space. A subset $A \subset X$
Thm: is ~~called~~ closed, if for any convergent sequence $x_1, x_2, \ldots \in A$, the limit is also in A.

[So "taking the limit" doesn't bring you out of A.]

Def: Let (X,d) be a metric space, $x \in X$ and $\emptyset \neq A \subset X$. The distance of the point x and the set A is

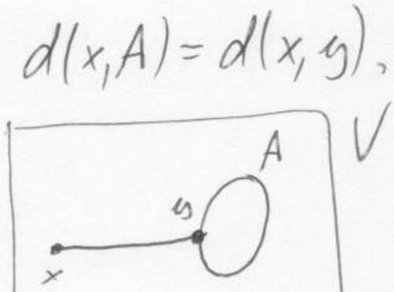
$$d(x,A) := \inf_{y \in A} d(x,y),$$

the "smallest possible" distance of any point in A from x.



Theorem (existence of nearest point):
Let V be a Hilbert space, $x \in V$ and $\emptyset \neq A \subset V$, A convex and closed.
Then $\exists!\, y \in A$ ~~st.~~ such that $d(x,A) = d(x,y)$.

[So: the the infimum in the definition of $d(x,A)$ is actually a unique minimum, y is the nearest point.]



Not surprising, but good to note:

Every assumption is important:
- V Hilbert
- A convex
- A closed

The proof is a good illustration of how geometry in a Hilbert space is like geometry on the ~~blackboard~~ blackboard.

Proof

Step 1 (easy practice): Assume $d(x,A)=0$. def. of $d(x,A)$

Then $\exists$ sequence $y_1, y_2, y_3 \ldots \in A$ s.t. $d(x,y_n) \longrightarrow 0$.

So $y_n \to x$, but A ~~was~~ is closed, so $x \in A$,

so $y := x$ will do (and no other point will).

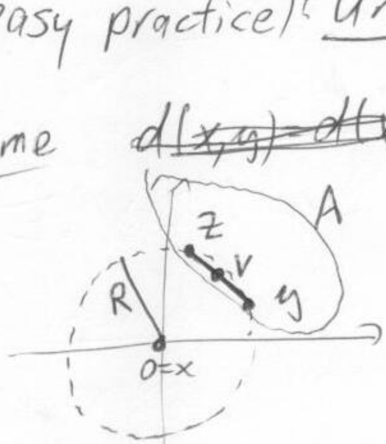
So from now on, assume $\boxed{R := d(x,A) > 0.}$

Step 2: w.l.o.g. assume $x = \underline{0}$. (makes formulas simpler)

Then $d(x,y) \overset{\text{simply}}{=\!=} \|y-x\| = \|y\| \quad \forall y \in V$.

Step 3: (easy practice): Uniqueness.

Assume ~~$d(x,y)=d(x,z)$~~ $d(0,y)=d(0,z)=R$, but $y \neq z$: and $y,z \in A$.

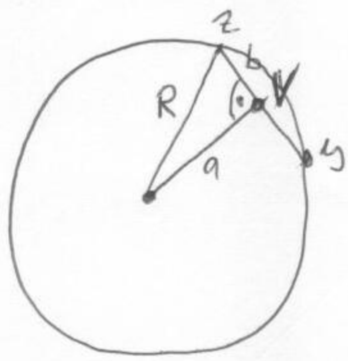


But A is convex, so $v := \frac{z+y}{2} \in A$ as well, and clearly $\|v\| < R$,

so $d(x,A) < R$, a contradiction.

["Clearly" means:

- see drawing for intuition
- write out using elementary 2-dim geometry



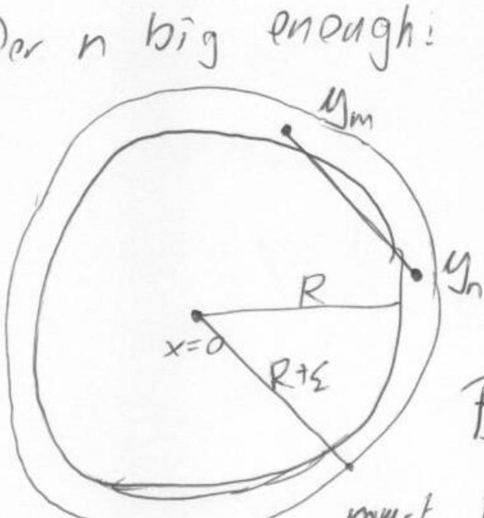
$R^2 = a^2 + b^2 > a^2$ if $b \neq 0$

translate this to scalar product notation]

Step 4: existence. Since $0 < R = d(x,A)$, there is a sequence $y_1, y_2 \ldots \in A$ s.t. $d(x, y_n) \to R$.

Claim: This sequence is Cauchy.

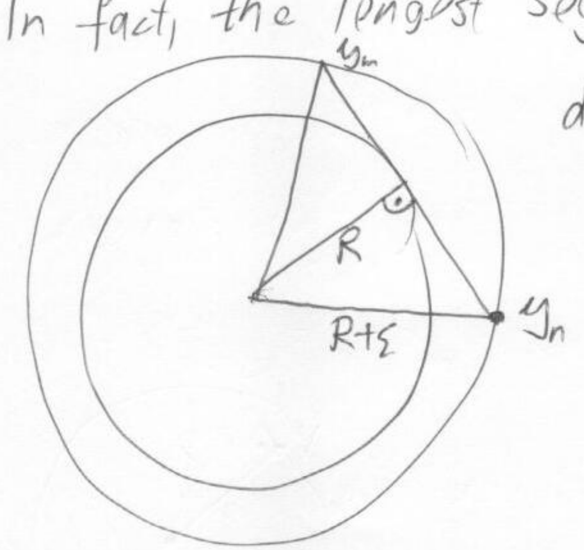
Indeed: $d(x, y_n) \to R$ means that $R \leq d(x, y_n) \leq R + \varepsilon$ for n big enough: so if n, m are big enough,



then both y_n and y_m are in the annulus $\{ v \in V \mid R \leq \|v\| \leq R + \varepsilon \}$

But A is convex, so y_n, y_m must be close, otherwise the segment connecting them gets closer than R to x, a contradiction.

In fact, the longest segment in the annulus:



$$d(y_n, y_m) \le 2\sqrt{(R+\varepsilon)^2 - R^2}$$

$$= 2\sqrt{2R\varepsilon + \varepsilon^2} \longrightarrow 0$$

as $\varepsilon \to 0$.

So $y_1, y_2, y_3 \ldots$ is Cauchy $\xRightarrow[\text{complete}]{V\text{ is}}$ $y := \lim_{n\to\infty} y_n$ exists $\Longrightarrow$

$\xRightarrow{A\text{ is closed}}$ $y \in A$.

Clearly $d(x,y) \le R+\varepsilon$ for every $\varepsilon > 0$, so $d(x,y) = R$ $\square$

Proof of the Riest represe~~tt~~tation thm

1.) Uniqueness: If $f: V \to \mathbb{R}$ is linear and $f(v) = \langle a, v\rangle = \langle b, v\rangle$ for all $v \in V$, then $a = b$, because choose $v = a-b$, so

easy

$$\langle a, a-b\rangle = f(a-b) = \langle b, a-b\rangle$$

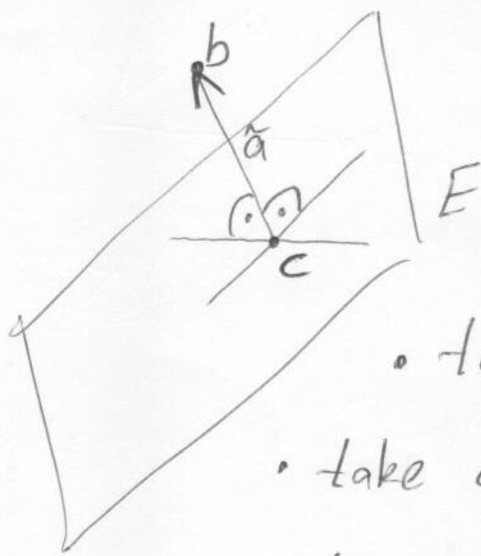
$$\Longrightarrow \langle a-b, a-b\rangle = \langle a, a-b\rangle - \langle b, a-b\rangle = f(a-b) - f(a-b) = 0 \checkmark$$

2.) Existence: GREAT IDEA: If f is really of the form $f(v) = \langle a, v\rangle$, then ~~the~~,

$$f(v) = 0 \iff \langle a, v\rangle = 0 \iff a \perp v, \text{ so}$$

the null space $E := \{v \in V \mid f(v) = 0\} \subset V$ is $\perp a$. So CONSTRUCT a to be $\perp E$.

[Actually, if the theorem is true, there will be only one vector $\perp E$ (up to constant multiple).]



GREAT IDEA 2:

To construct the normal vector:

- take any $b \notin E$
- take $c \in E$ which is $\boxed{\text{nearest}}$ to b
- $a := b - c$ will do (up to constant factor).

Some details:

- If $f(v) \equiv 0$, then $a := 0$ will do, so assume $f \not\equiv 0$.

Then $\exists\, b_0 \in V$ s.t. $f(b_0) \neq 0$. ~~Let $b = \frac{b_0}{f(b_0)}$, so $f(b) = 1$.~~

$$\text{Let } b = \frac{b_0}{f(b_0)}, \text{ so } f(b) \overset{\text{linearity}}{=\!=} 1.$$

Set $E := \{v \in V \mid f(v) = 0\}$, the null space of f.

E is clearly convex. In fact E is a linear subspace, so if $x, y \in V$, $\alpha, \beta \in \mathbb{R}$, then $\alpha x + \beta y \in E$, so any linear combination is in E, not only convex ones.

Proof: $f(\alpha x + \beta y) \overset{\text{linearity}}{=\!=} \alpha f(x) + \beta f(y) = \alpha \cdot 0 + \beta \cdot 0 = 0$ ✓

<u>Claim:</u> E is <u>closed</u>.

<u>Proof:</u> We know that $f: V \to \mathbb{R}$ is ~~cont~~ bounded $\Rightarrow$

$\Rightarrow$ <u>continuous</u>,

so if $x_1, x_2 \ldots \in E$ and $x_n \to x$,

then $f(x_n) \to f(x)$, but $x_n \in E$ means $f(x_n) = 0$,

so $f(x) = 0$, so $x \in E$ ✓

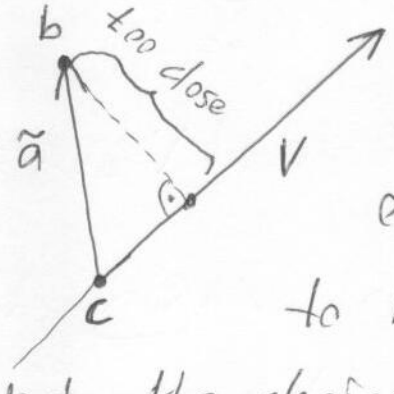
~~Now use the~~

Since $E \subset V$ is convex and closed, <u>use the previous theorem</u> to get $c \in E$ with $d(b, E) = d(b, c)$,

and set $\tilde{a} := b - c$.

<u>Claim:</u> $\tilde{a} \perp E$, meaning $\langle \tilde{a}, v \rangle = 0$ for all $v \in E$

<u>Proof:</u>



If not, then the line crossing c in the direction of v will get <u>closer</u> ~~than~~ to b than c was. This contradicts the choice of c. <u>HW.</u> ✓

In particular, $c \in E$, so $\langle \tilde{a}, c \rangle = 0$.

Now $f(b) = 1$,

while $\langle \tilde{a}, b \rangle = \langle \tilde{a}, c + \tilde{a} \rangle = \underbrace{\langle \tilde{a}, c \rangle}_{0} + \langle \tilde{a}, \tilde{a} \rangle =: \lambda$

So ~~set~~ LET $\boxed{a = \frac{\tilde{a}}{\lambda}}$ $\lambda = \langle \tilde{a}, b \rangle$

$f(b) = 1$

Claim: this works - HW.

Hint: if $\underbrace{v - \mu \cdot \tilde{a}}_{\text{projection of } v \text{ on } E \text{ in the direction of } \tilde{a}} \in E$ with suitable $\mu \in \mathbb{R}$, then $\langle v - \mu\tilde{a}, \tilde{a} \rangle = 0$.

$\square$