Rotation invariance means that you see the same function along any radial line, including the axes.

E.g. choosing $y=0$ we have

$$f(x,0)=g(x)=h(x)\,h(0)$$

so g and h are essentially the same: $g(x)=c\,h(x)$

with $c:=h(0)$

If $c:=h(0)=0$, then ~~g~~ $g(x)=c\,h(x)=0\ \forall x$, so

$\underline{\underline{f\equiv 0}}$, which is clearly a solution.

Now assume $c=h(0)\neq 0$. Then we can „scale down" with it:

let $\tilde{h}=\frac{h}{c}$, $\tilde{g}=\frac{g}{c^2}$. Then $\underline{\underline{\tilde{h}=\tilde{g}}}$, and

substituting back gives $\cancel{c}\,\tilde{g}(\sqrt{x^2+y^2})=\cancel{c}\,\tilde{g}(x)\,\cancel{c}\,\tilde{g}(y)$.

Now use the hint and introduce $u:=x^2$, $v:=y^2$,

$\tilde{G}(u):=\tilde{g}(\sqrt{u})$, so $\tilde{g}(x)=\tilde{G}(x^2)$ to get

$$\boxed{\tilde{G}(u+v)=\tilde{G}(u)\,\tilde{G}(v)}$$

TAKE THE LOGARITHM

! $\gamma(u)=\ln\tilde{G}(u)$, so $\gamma(u+v)=\gamma(u)+\gamma(v)$, $\boxed{\gamma \text{ is additive}}$!

We assumed that f (and g) were continuous,

and a continuous, additive function is $\boxed{\text{linear}}$

[Proof: choosing $u=v$, we got $\gamma(2u)=2\gamma(u)$, which implies that $\gamma(q)=q\,\gamma(1)$ for every q of the form $q=2^{-n}$. This in turn implies $\gamma(r)=r\,\gamma(1)$ for all $r=k2^{-n}$. These r are dense so $\gamma(x)=\gamma(1)\cdot x$ for all x by continuity]

So choosing $B := \gamma(1)$ we get $\gamma(u) = Bu$

$$\Rightarrow \tilde{G}(u) = e^{\gamma(u)} = e^{Bu}$$

$$\cancel{\tilde{h}(x) \Rightarrow \tilde{g}(x) = \tilde{G}(x^2) = e^{Bx^2}}$$

$$\Rightarrow \tilde{h}(x) = \tilde{g}(x) = \tilde{G}(x^2) = e^{Bx^2}$$, So chose $A := c^2 = h(0)^2$

to get

$$\boxed{\boxed{f(x,y) = A e^{B(x^2+y^2)}} \text{ with } A \geq 0,\ B \in \mathbb{R}}$$

Remark: If you are worried about $\tilde{G}(u)$ being zero when taking the logarithm $\gamma(u) := \ln \tilde{G}(u)$,

use $\tilde{G}(u+v) = \tilde{G}(u)\tilde{G}(v)$ directly with $u=v$ to

get $\tilde{G}(2u) = [G(u)]^2$ or $\tilde{G}(u) = G\left(\left(\frac{u}{2}\right)^2\right)$

which implies that if $\tilde{G}(1) \neq 0$, then $\tilde{G}(r) > 0$

for all $r = k e^{-2n}$. This implies $\tilde{G}(u) > 0$ for $u > 0$

by continuity.