Let $I = \int_{(0,\infty)^2} f(x,y)\,dx\,dy$

a.) $I = \int_0^\infty x^{s-1} e^{-x}\,dx \int_0^\infty y^{t-1} e^{-y}\,dy \stackrel{def}{=} \Gamma(s)\Gamma(t)$

b.) Let $u = x+y$, $\xi = \frac{y}{x+y}$, so $\boxed{y = u\xi}$ and $\boxed{x = u(1-\xi),}$

and $(x,y) \in (0,\infty)\times(0,\infty)$ corresponds to $(u,\xi)\in(0,\infty)\times(0,1)$.

Now the Jacobian of the substitution is

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \xi} \end{pmatrix} = \begin{pmatrix} 1-\xi & -u \\ \xi & u \end{pmatrix} \Rightarrow \underline{\underline{|\det J|}} = |(1-\xi)u - (-u\xi)| = \underline{\underline{u}}$$

$$\Rightarrow I = \int_0^\infty \int_0^1 [u(1-\xi)]^{s-1} e^{-u(1-\xi)} (u\xi)^{t-1} e^{-u\xi}\, u \, d\xi\, du =$$

$$= \int_0^\infty \int_0^1 u^{s-1+t-1+1} e^{-u} \cancel{e^{u\xi}} \cancel{e^{-u\xi}} \xi^{t-1} (1-\xi)^{s-1}\, d\xi\, du =$$

$$= \int_0^\infty u^{s+t-1} e^{-u}\, du \int_0^1 (1-\xi)^{s-1} \xi^{t-1}\, d\xi \stackrel{def}{=} \Gamma(s+t)B(s,t).$$

Comparing the two gives $B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$