$$I_n = \int_{-1}^{1} e^{nf(x)}\,dx \quad \text{with} \quad f(x) = \ln\sqrt{1-x^2} = \frac{1}{2}\ln(1-x^2)$$

so $f:(-1,1) \to \mathbb{R}$ is twice differentiable with a unique global maximum at $x_0 = 0$:

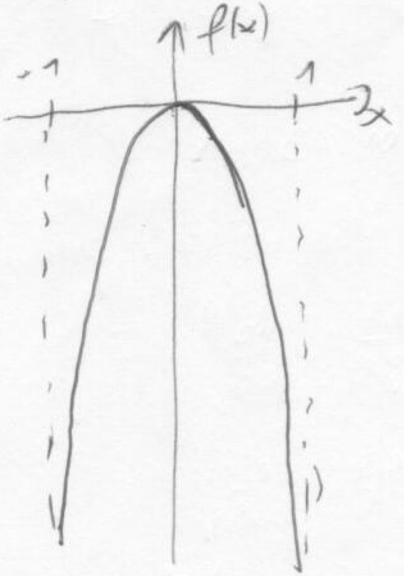
$$f'(x) = \frac{1}{2}\,\frac{-2x}{1-x^2} = \frac{-x}{1-x^2} \qquad = 0 \quad \text{iff} \quad x = 0$$

$$f''(x) = \frac{-(1-x^2)-(-x)(-2x)}{(1-x^2)^2} = \frac{-1+x^2-2x^2}{(1-x^2)^2} = -\frac{1+x^2}{(1-x^2)^2}$$

$$f''(0) = -1$$

So let $A := f(0) = \frac{1}{2}\ln 1 = 0, \quad B = -f''(0) = 1$

Actually f is not defined at -1 and $+1$, ~~so it~~ and it can not be continuously extended to $[-1,1]$:



so the theorem about almost-Gaussian integrals only applies in its generalized form. But luckily, f is separated away from $A = f(0)$ when x is not close to 0, so the therem gives

$$I_n = \int_{-1}^{1} e^{nf(x)}\,dx \sim e^{nA}\sqrt{\frac{2\pi}{nB}} \;\underset{B=1}{\overset{A=0}{=}}\; \sqrt{\frac{2\pi}{n}}$$