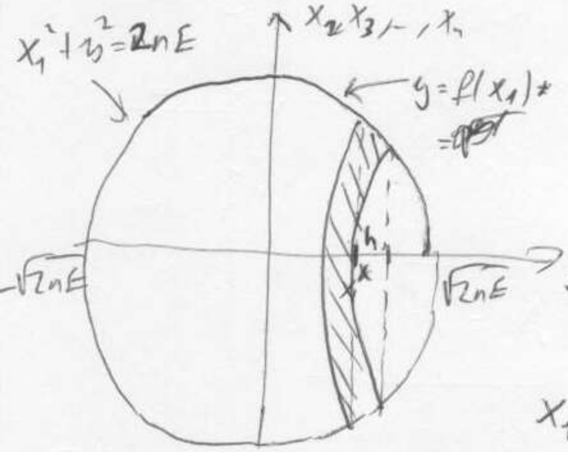
Let h be small and look at $\mathbb{P}(x \le V_1 \le x+h) \approx h f_n(x)$.



The event $\{x \le V_1 \le x+h\}$ means that $V=(V_1, V_2, \ldots, V_n)$ (which is on the sphere) is on the narrow strip between $x_1 = x$ and $x_1 = x+h$.

Since V is uniformly distributed on the sphere,

$$\mathbb{P}(x \le V_1 \le x+h) = \frac{\text{surface (strip)}}{\text{surface (sphere)}} \quad \left[\begin{array}{c}\text{for} \\ -\sqrt{2nE} < x < \sqrt{2nE}\end{array}\right]$$

Since we are in $\mathbb{R}^n$, "surface" means $(n-1)$-dimensional surface volume.

Since the stip is narrow, ~~Surface(strip) ≈~~

$$\text{surface (strip)} \approx \text{width} \cdot \text{length}$$

but here "length" means $\boxed{(n-2)\text{-dimensional}}$ surface volume of the $\boxed{1 \text{ less dimensional sphere}}$ at $\{x_1 = x\}$.

From the graph above: radius $= \sqrt{2nE - x^2} = f(x)$

$$\text{width} = \sqrt{1+f'^2(x)}\, h = \sqrt{1+\left(\frac{-\cancel{2}x}{\cancel{2}\sqrt{2nE-x^2}}\right)^2}\, h = \sqrt{\frac{2nE-\cancel{x^2}+\cancel{x^2}}{2nE-x^2}}\, h$$

We had the notation C_n for the surface of the unit sphere in $\mathbb{R}^n$, so

So

$$h\,f_n(x) \approx \mathbb{P}(x \le V_1 \le x+h) \approx \frac{h\sqrt{\frac{2nE}{2nE-x^2}}\;C_{n-1}\sqrt{2nE-x^2}^{\,n-2}}{C_n\sqrt{2nE}^{\,n-1}}$$

dimension

~~[illegible]~~ dimension

[More precisely: $\mathbb{P}(x \le V_1 \le x+h) = h\,f_n(x) + o(h)$

and $\mathbb{P}(x \le V_1 \le x+h) = \quad + o(h)$]

So
$$f_n(x) = \frac{\sqrt{\frac{2nE}{2nE-x^2}}\;C_{n-1}\sqrt{2nE-x^2}^{\,n-2}}{C_n\sqrt{2nE}^{\,n-1}}$$

$$= \frac{C_{n-1}}{C_n}\frac{1}{\sqrt{2nE}}\left(1-\frac{x^2}{2nE}\right)^{\frac{n-3}{2}} = A_n\left(1-\frac{x^2}{2nE}\right)^{\frac{n-3}{2}}$$

with $A_n = \frac{C_{n-1}}{C_n\sqrt{2nE}}$. Of course, $A_n = \dfrac{1}{\int_{-\sqrt{2nE}}^{\sqrt{2nE}}\left(1-\frac{x^2}{2nE}\right)^{\frac{n-3}{2}}}$

More precisely: $\boxed{f_n(x) = \begin{cases} A_n\left(1-\frac{x^2}{2nE}\right)^{\frac{n-3}{2}}, & \text{if } -\sqrt{2nE} < x < \sqrt{2nE} \\ 0, & \text{if not} \end{cases}}$

<u>Limit:</u> $\left(1-\frac{x^2}{2nE}\right)^{\frac{n-3}{2}} \xrightarrow{n\to\infty} e^{-\frac{x^2}{4E}}$, so $f_n(x) \to \boxed{f(x) = A\,e^{-\frac{x^2}{4E}}}$

where $A := \lim_{n\to\infty} A_n$. It's not surprising (we could check) that $\int_{-\infty}^{\infty} f(x)\,dx = 1$, so $A = \frac{1}{\sqrt{2\pi}\sqrt{2E}}$.

$\boxed{\text{f is the Gaussian density with } m = 0,\ \sigma = \sqrt{2E}}$