a) i.) Let $B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$ for $i = 2, 3, \dots$

Then $A_n = B_1 \cup B_2 \cup \dots \cup B_n$ <u>disjoint</u> union

and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$, so by σ-additivity and additivity

$$\mu(A_n) = \sum_{i=1}^{n} \mu(B_i)$$

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(B_i) \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu(A_n) \quad \square$$

ii.) Let $\tilde{A}_i = A_1 \setminus A_i$, so $\tilde{A}_i$ is an <u>increasing</u> sequence, and the previous part gives

$$\mu\Big(A_1 \setminus \bigcap_{i=1}^{\infty} A_i\Big) = \mu\Big(\bigcup_{i=1}^{\infty} \tilde{A}_i\Big) = \cancel{\sum_{i=1} \lim_{n\to\infty} \mu(\tilde{A}_n)}$$

$$= \lim_{n\to\infty} \mu(\tilde{A}_n)$$

Now by additivity

$$\mu(A_1) = \mu\Big(A_1 \setminus \bigcap_{i=1}^{\infty} A_i\Big) + \mu\Big(\bigcap_{i=1}^{\infty} A_i\Big)$$

and

$$\mu(A_1) = \mu(\tilde{A}_n) + \mu(A_n)$$

, so $\boxed{\text{If } \mu(A_1) < \infty}$

then $\mu(A_n) \to \mu\Big(\bigcap_{i=1}^{\infty} A_i\Big) \quad \square$

b.) Let $\Omega = \mathbb{R}$, $\mu = \text{Leb}$, $A_n = [n, \infty)$.

Then $\mu\Big(\bigcap_n A_n\Big) = \mu(\emptyset) = 0$, but $\mu(A_n) = \infty$ for $\forall n$.